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CAUCHY RESIDUES AND DE RHAM HOMOLOGY

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This paper represents my third attempt to write up a suitable generalization of the classical Cauchy Residue theorem. As I pushed the earlier versions for naturality and generality I was ultimately lead to a new foundation of de Rham homology free of the theory of distributions but relying on basic sheaf theory much like the Borel Moore homology theory.

Singular homology and de Rham homology agree on a smooth manifold. The whole point in introducing de Rham homology is the posibility of alternative representations of homology classes. This is amply illustrated by the general Cauchy residue formula given at the end of the paper.

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1. COMPACT CHAINS

Let X denote a smooth n-dimensional manifold. For an integer p we let $\Gamma(X, \Omega^p)$ denote the space of $\mathbf{C}$-valued differential p-forms on X. By a *compact p-chain* on X we understand a $\mathbf{C}$-valued linear form T on $\Gamma(X, \Omega^p)$ for which there exists a compact subset K of X such that

$$(1.1) \qquad < T, \omega > \; = 0 \,, \quad \omega \in \Gamma(X, \Omega^p) \,, \quad \operatorname{Supp}(\omega) \cap K = \emptyset \,,$$

where the bracket denotes simple evaluation of a linear form. The compact p-chain T on X gives rise to a $(p-1)$-chain bT on X given by

$$(1.2) \qquad < bT, \omega > \; = \; < T, d\omega > \,, \quad \omega \in \Gamma(X, \Omega^{p-1}) \,.$$

The operator b makes the compact p-chains on X into a complex, which we denote $D^c_\bullet(X, \mathbf{C})$. A compact p-chain T is *closed* if $bT = 0$. By a compact *p-cycle* we understand a closed p-chain, while a compact *p-boundary* is a p-cycle of the form bW, where W is a compact $(p+1)$-chain. We say that p-cycles S and T are *homologous* if $T - S$ is a p-boundary. An explicit relation of the form

$$T - S = bW, \quad W \in D^c_{p+1}(X, \mathbf{C}), \tag{1.3}$$

is called a *de Rham homology* from S to T. Explicit de Rham homologies are often constructed on the basis of Stokes theorem, compare the formulas to the right of the drawings in section 7.

Let us make the important observation that homologous p-chains have the same evolution on any closed p-form. The group of de Rham homology classes is denoted by

$$H^c_p(X, \mathbf{C}) = H_p D^c_{\bullet}(X, \mathbf{C}). \tag{1.4}$$

The letter c in the homology symbol is borrowed from Haefliger's exposé in [1].

A smooth map $f: X \to Y$ will induce a chain map from the complex of compact chains on X to the complex of compact chains on Y

$$f_*: D^c_{\bullet}(X, \mathbf{C}) \to D^c_{\bullet}(Y, \mathbf{C}).$$

To see this notice that a given compact subset K on X and a p-form ω on Y supported outside $f(K)$ pulls back to a form $f^*\omega$ supported outside K. We can now define f_*T by the formula

$$\langle f_*T, \omega \rangle = \langle T, f^*\omega \rangle, \quad T \in D^c_p(X, \mathbf{C}), \omega \in \Gamma(Y, \Omega^p). \tag{1.5}$$

These remarks make it clear how to turn de Rham homology into a covariant functor on the smooth category.

Zero cycles. Evaluation of a compact zero cycle Z against the constant function 1 defines the *degree of the zero cycle*

$$\mathrm{dg}(Z) = \langle Z, 1 \rangle, \quad Z \in D^c_0(X, \mathbf{C}). \tag{1.6}$$

A point $x \in X$ defines a zero cycle of degree 1, the *Dirac 0-cycle* δ_x given by

$$\langle \delta_x, \varphi \rangle = \varphi(x), \quad \varphi \in \Gamma(X, \Omega^0). \tag{1.7}$$

A continuously differentiable curve $\gamma: [a, b] \to X$ with endpoints $x = \gamma(a)$ and $y = \gamma(b)$ defines a de Rham homology from δ_x to δ_y:

$$\int_\gamma d\varphi = \varphi(y) - \varphi(x), \quad \varphi \in \Gamma(X, \Omega^0). \tag{1.8}$$

In case X is connected, then all zero cycles of degree zero are homologous to zero as it follows from the result of the next section.

A smooth map $f : X \to Y$ will preserve the degree of a zero cycle in the sense that

$$\mathrm{dg}\,(f_* T) = \mathrm{dg}\, T\,, \quad T \in D_0^c(X, \mathbf{C})\,, \tag{1.9}$$

as it follows from (1.5).

The reader is invited to replace $\mathbf{C}$ by $\mathbf{R}$ and change the meaning of the symbol Ω from complex to real differential forms.

2. Biduality

In this section we shall show that de Rham cohomology can be calculated as the linear dual of de Rham homology in the same way singular cohomology can be obtained from singular homology.

(2.1) Theorem. *Let X denote a smooth manifold. Evaluation of a compact p-chain against a p-form induces an isomorphism*

$$H^p(X, \mathbf{C}) \tilde{\to} \mathrm{Hom}\,(H_p^c(X, \mathbf{C}), \mathbf{C})$$

for all integers p.

Proof. The hart of the matter is of sheaf theoretic nature, so we start with a brief review during which the reader is invited to change the meaning of the letter X to denote a general locally compact space and the letter $\mathbf{C}$ to denote an arbitrary field. For notation and details the reader may consult [5] V.1, and the references given there.

To a *soft* $\mathbf{C}$-sheaf F on X we can associate the sheaf $F^\vee$ whose sections over the open subset U of X are given by

$$\Gamma(U, F^\vee) = \mathrm{Hom}\,(\Gamma_c(U, F), \mathbf{C}) \tag{2.2}$$

Restriction in the sheaf $F^\vee$ from U to a smaller open subset V is the $\mathbf{C}$-linear dual of "extension by zero"

$$\Gamma_c(V, F) \to \Gamma_c(U, F)\,, \quad V \subseteq U\,.$$

The presheaf $F^\vee$ we have described is actually a sheaf and indeed a *soft* sheaf. This allows us to iterate the construction and form $F^{\vee\vee}$. We shall construct a natural *biduality morphism* of $\mathbf{C}$-sheaves on X

$$F \to F^{\vee\vee}\,. \tag{2.3}$$