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Artikel: ISOCLINIC n -PLANES IN \mathbb{R}^{2n} AND THE HOPF-STEENROD SPHERE BUNDLES $S^{2n-1} \rightarrow S^n, \text{quad } n=2,4,8$
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- (ii) not commutative, i.e., generally, $XY \neq YX$ (but see (4) (iv) below);
 - (iii) not associative, i.e., generally, $(XY)W \neq X(YW)$ (but see (7) below).
- (4) The *real part* of $X \equiv (q_1, q_2)$ is $\text{Re } X = (\text{Re } q_1, 0) \equiv \text{Re } q_1$. X is said to be *real* if $X = \text{Re } X$; i.e., (q_1, q_2) is real iff q_1 is real and $q_2 = 0$.
- (i) $\text{Re}(X + Y) = \text{Re}(X) + \text{Re}(Y)$.
 - (ii) $\text{Re}(XY) = \text{Re}(YX)$.
 - (iii) $\text{Re}(CX) = 0$ for all X implies that $C = 0$.
 - (iv) $CX = XC$ for all X iff C is real. In this case, $C = (c_1, 0)$, where $c_1 = \text{real}$, and $CX = (c_1q_1, c_1q_2) = XC$.
- (5) The *conjugate* of $X \equiv (q_1, q_2)$ is $X^* = (q_1^*, -q_2)$.
- (i) $(X + Y)^* = X^* + Y^*$,
 - (ii) $(XY)^* = Y^*X^*$.
 - (iii) $X^* = X$ iff X is real.
- (6) The *norm* of X is the non-negative real number $N(X) \equiv XX^*$, which is also equal to X^*X . The *length* of X is the non-negative real number $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$.
- (i) $N(X) = 0$ iff $X = 0$.
 - (ii) If $X \neq 0$, then $X^{-1} \equiv X^*/N(X)$ is a right and left inverse of X .
 - (iii) $N(XY) = N(X)N(Y)$. It follows from this that $XY = 0$ iff $X = 0$ or $Y = 0$.
- (7) Though multiplication is generally non-associative,
- (i) $(XY)Y^* = X(Y Y^*)$.
 - (ii) If $Y \neq 0$, then $(XY)Y^{-1} = X = Y^{-1}(YX)$.
 - (iii) $\text{Re}((XY)W) = \text{Re}(X(YW))$.

APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of S^{2n-1} by S^{n-1} over S^n , $n = 2, 4$, or 8 , by intersecting the unit sphere S^{2n-1} in $R^{2n} = Q_n \times Q_n$ with the Q_n -lines $Y = CX$ and $X = 0$. In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic n -planes in R^{2n} are equivalent concepts. Here we prove, directly, the

THEOREM A2.1. *The set of Q_n -lines $\{Y = CX, X = 0\}$ in $Q_n \times Q_n$, when viewed as n -planes in R^{2n} , are mutually isoclinic n -planes.*

Proof. We shall prove the theorem for the case $n = 8$ only. The proof for the cases $n = 2, 4$ follows the same line and is simpler.

Some preliminaries are necessary. Suppose that under the identification of $Q_8 \times Q_8$ with R^{16} as in Theorem 5.1, the elements $(X, Y), (X', Y')$ of $Q_8 \times Q_8$ become the vectors $(X, Y), (X', Y')$ in R^{16} with respectively the components $(x_1, \dots, x_{16}), (x'_1, \dots, x'_{16})$. Then it can easily be verified that the inner product of the two vectors (X, Y) and (X', Y') is

$$\langle (X, Y), (X', Y') \rangle \equiv \sum_{i=1}^{16} x_i x'_i = \operatorname{Re} (XX'^* + YY'^*).$$

It follows from this that the length of the vector (X, Y) is

$$|(X, Y)| = \langle (X, Y), (X, Y) \rangle^{1/2} = (XX^* + YY^*)^{1/2},$$

and that the two vectors (X, Y) and (X', Y') are orthogonal if and only if $\operatorname{Re} (XX'^* + YY'^*) = 0$.

We can now prove our theorem by showing that in R^{16} , the 8-plane $\mathbf{A}: Y = AX$ is isoclinic with the 8-planes $\mathbf{B}: Y = BX$ and $\mathbf{O}^\perp: X = 0$.

Let $(T, BT) \in \mathbf{B}$ be the projection of any nonzero vector $(X, AX) \in \mathbf{A}$ on \mathbf{B} . Then the vector $(X - T, AX - BT)$ is orthogonal to \mathbf{B} , i.e., it is orthogonal to all the vectors $(W, BW) \in \mathbf{B}$, where W is an arbitrary Cayley number. Therefore,

$$(A.1) \quad \operatorname{Re} \{(X - T)W^* + (AX - BT)(BW)^*\} = 0 \quad \text{for all } W \in Q_8.$$

Since, by (4) (ii) and (7) (iii) in Appendix 1, the terms inside the brackets in $\operatorname{Re} \{ \}$ are commutative and associative, the left-hand side of (A.1) is equal to

$$\begin{aligned} & \operatorname{Re} \{(X - T)W^* + [(AX - BT)W^*]B^*\} \\ &= \operatorname{Re} \{(X - T)W^* + [B^*(AX - BT)]W^*\} \\ &= \operatorname{Re} \{(X - T)W^* + [(B^*A)X - (B^*B)T]W^*\} \\ &= \operatorname{Re} \{[X - T + (B^*A)X - (B^*B)T]W^*\}. \end{aligned}$$

Therefore, by (4) (iii) in Appendix 1, condition (A.1) implies that

$$X - T + (B^*A)X - (B^*B)T = 0,$$

and hence

$$(A.2) \quad T = (1 + B^*A)X / (1 + B^*B).$$

Now, the squared length of the vector (X, AX) is

$$\begin{aligned} |(X, AX)|^2 &= XX^* + (AX)(AX)^* \\ &= XX^* + AA^*XX^*, \end{aligned}$$

i.e.,

$$(A.3) \quad |(X, AX)|^2 = (1 + A^*A)XX^*.$$

Similarly,

$$|(T, BT)|^2 = (1 + B^*B)TT^*.$$

But by (A.2),

$$\begin{aligned} TT^* &= (1 + B^*A)X[(1 + B^*A)X]^*/(1 + B^*B)^2 \\ &= (1 + B^*A)(1 + A^*B)XX^*/(1 + B^*B)^2. \end{aligned}$$

Therefore,

$$(A.4) \quad |(T, BT)|^2 = (1 + B^*A)(1 + A^*B)XX^*/(1 + B^*B).$$

Hence, it follows from (A.3) and (A.4) that the angle θ between the vector $(X, AX) \in \mathbf{A}$ and its projection on \mathbf{B} is given by

$$\cos^2\theta = \frac{|(T, BT)|^2}{|(X, AX)|^2} = \frac{(1 + A^*B)(1 + B^*A)}{(1 + A^*A)(1 + B^*B)},$$

which shows that the angle between any nonzero vector $(X, AX) \in \mathbf{A}$ and its projection on \mathbf{B} is independent of the choice of X ; that is, the 8-plane \mathbf{A} is isoclinic with the 8-plane \mathbf{B} .

Finally, to show that the 8-plane $\mathbf{A}: Y = AX$ is isoclinic with the 8-plane $\mathbf{O}^\perp: X = 0$, we need only observe that the projection of the nonzero vector $(X, AX) \in \mathbf{A}$ on \mathbf{O}^\perp is the vector (O, AX) , and

$$\frac{|(O, AX)|^2}{|(X, AX)|^2} = \frac{(AX)(AX)^*}{(1 + A^*A)XX^*} = \frac{AA^*}{1 + AA^*}$$

is independent of X .