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**Artikel:** ISOCLINIC  $n$ -PLANES IN  $S^{2n}$  AND THE HOPF-STEENROD SPHERE BUNDLES  $S^{2n-1} \rightarrow S^n, \text{quad } n=2,4,8$   
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- (ii) not commutative, i.e., generally,  $XY \neq YX$  (but see (4) (iv) below);
- (iii) not associative, i.e., generally,  $(XY)W \neq X(YW)$  (but see (7) below).
- (4) The *real part* of  $X \equiv (q_1, q_2)$  is  $\text{Re } X = (\text{Re } q_1, 0) \equiv \text{Re } q_1$ .  $X$  is said to be *real* if  $X = \text{Re } X$ ; i.e.,  $(q_1, q_2)$  is real iff  $q_1$  is real and  $q_2 = 0$ .
- (i)  $\text{Re}(X + Y) = \text{Re}(X) + \text{Re}(Y)$ .
- (ii)  $\text{Re}(XY) = \text{Re}(YX)$ .
- (iii)  $\text{Re}(CX) = 0$  for all  $X$  implies that  $C = 0$ .
- (iv)  $CX = XC$  for all  $X$  iff  $C$  is real. In this case,  $C = (c_1, 0)$ , where  $c_1 = \text{real}$ , and  $CX = (c_1q_1, c_1q_2) = XC$ .
- (5) The *conjugate* of  $X \equiv (q_1, q_2)$  is  $X^* = (q_1^*, -q_2)$ .
- (i)  $(X + Y)^* = X^* + Y^*$ ,
- (ii)  $(XY)^* = Y^*X^*$ .
- (iii)  $X^* = X$  iff  $X$  is real.
- (6) The *norm* of  $X$  is the non-negative real number  $N(X) \equiv XX^*$ , which is also equal to  $X^*X$ . The *length* of  $X$  is the non-negative real number  $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$ .
- (i)  $N(X) = 0$  iff  $X = 0$ .
- (ii) If  $X \neq 0$ , then  $X^{-1} \equiv X^*/N(X)$  is a right and left inverse of  $X$ .
- (iii)  $N(XY) = N(X)N(Y)$ . It follows from this that  $XY = 0$  iff  $X = 0$  or  $Y = 0$ .
- (7) Though multiplication is generally non-associative,
- (i)  $(XY)Y^* = X(Y Y^*)$ .
- (ii) If  $Y \neq 0$ , then  $(XY)Y^{-1} = X = Y^{-1}(YX)$ .
- (iii)  $\text{Re}((XY)W) = \text{Re}(X(YW))$ .

## APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of  $S^{2n-1}$  by  $S^{n-1}$  over  $S^n$ ,  $n = 2, 4$ , or  $8$ , by intersecting the unit sphere  $S^{2n-1}$  in  $R^{2n} = Q_n \times Q_n$  with the  $Q_n$ -lines  $Y = CX$  and  $X = 0$ . In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic  $n$ -planes in  $R^{2n}$  are equivalent concepts. Here we prove, directly, the

**THEOREM A2.1.** *The set of  $Q_n$ -lines  $\{Y = CX, X = 0\}$  in  $Q_n \times Q_n$ , when viewed as  $n$ -planes in  $R^{2n}$ , are mutually isoclinic  $n$ -planes.*

*Proof.* We shall prove the theorem for the case  $n = 8$  only. The proof for the cases  $n = 2, 4$  follows the same line and is simpler.

Some preliminaries are necessary. Suppose that under the identification of  $Q_8 \times Q_8$  with  $R^{16}$  as in Theorem 5.1, the elements  $(X, Y), (X', Y')$  of  $Q_8 \times Q_8$  become the vectors  $(X, Y), (X', Y')$  in  $R^{16}$  with respectively the components  $(x_1, \dots, x_{16}), (x'_1, \dots, x'_{16})$ . Then it can easily be verified that the inner product of the two vectors  $(X, Y)$  and  $(X', Y')$  is

$$\langle (X, Y), (X', Y') \rangle \equiv \sum_{i=1}^{16} x_i x'_i = \operatorname{Re} (XX'^* + YY'^*).$$

It follows from this that the length of the vector  $(X, Y)$  is

$$|(X, Y)| = \langle (X, Y), (X, Y) \rangle^{1/2} = (XX^* + YY^*)^{1/2},$$

and that the two vectors  $(X, Y)$  and  $(X', Y')$  are orthogonal if and only if  $\operatorname{Re} (XX'^* + YY'^*) = 0$ .

We can now prove our theorem by showing that in  $R^{16}$ , the 8-plane  $\mathbf{A}: Y = AX$  is isoclinic with the 8-planes  $\mathbf{B}: Y = BX$  and  $\mathbf{O}^\perp: X = 0$ .

Let  $(T, BT) \in \mathbf{B}$  be the projection of any nonzero vector  $(X, AX) \in \mathbf{A}$  on  $\mathbf{B}$ . Then the vector  $(X - T, AX - BT)$  is orthogonal to  $\mathbf{B}$ , i.e., it is orthogonal to all the vectors  $(W, BW) \in \mathbf{B}$ , where  $W$  is an arbitrary Cayley number. Therefore,

$$(A.1) \quad \operatorname{Re} \{(X - T)W^* + (AX - BT)(BW)^*\} = 0 \quad \text{for all } W \in Q_8.$$

Since, by (4) (ii) and (7) (iii) in Appendix 1, the terms inside the brackets in  $\operatorname{Re} \{ \quad \}$  are commutative and associative, the left-hand side of (A.1) is equal to

$$\begin{aligned} & \operatorname{Re} \{(X - T)W^* + [(AX - BT)W^*]B^*\} \\ &= \operatorname{Re} \{(X - T)W^* + [B^*(AX - BT)]W^*\} \\ &= \operatorname{Re} \{(X - T)W^* + [(B^*A)X - (B^*B)T]W^*\} \\ &= \operatorname{Re} \{[X - T + (B^*A)X - (B^*B)T]W^*\}. \end{aligned}$$

Therefore, by (4) (iii) in Appendix 1, condition (A.1) implies that

$$X - T + (B^*A)X - (B^*B)T = 0,$$

and hence

$$(A.2) \quad T = (1 + B^*A)X / (1 + B^*B).$$

Now, the squared length of the vector  $(X, AX)$  is

$$\begin{aligned} |(X, AX)|^2 &= XX^* + (AX)(AX)^* \\ &= XX^* + AA^*XX^*, \end{aligned}$$

i.e.,

$$(A.3) \quad |(X, AX)|^2 = (1 + A^*A)XX^*.$$

Similarly,

$$|(T, BT)|^2 = (1 + B^*B)TT^*.$$

But by (A.2),

$$\begin{aligned} TT^* &= (1 + B^*A)X[(1 + B^*A)X]^*/(1 + B^*B)^2 \\ &= (1 + B^*A)(1 + A^*B)XX^*/(1 + B^*B)^2. \end{aligned}$$

Therefore,

$$(A.4) \quad |(T, BT)|^2 = (1 + B^*A)(1 + A^*B)XX^*/(1 + B^*B).$$

Hence, it follows from (A.3) and (A.4) that the angle  $\theta$  between the vector  $(X, AX) \in \mathbf{A}$  and its projection on  $\mathbf{B}$  is given by

$$\cos^2\theta = \frac{|(T, BT)|^2}{|(X, AX)|^2} = \frac{(1 + A^*B)(1 + B^*A)}{(1 + A^*A)(1 + B^*B)},$$

which shows that the angle between any nonzero vector  $(X, AX) \in \mathbf{A}$  and its projection on  $\mathbf{B}$  is independent of the choice of  $X$ ; that is, the 8-plane  $\mathbf{A}$  is isoclinic with the 8-plane  $\mathbf{B}$ .

Finally, to show that the 8-plane  $\mathbf{A}: Y = AX$  is isoclinic with the 8-plane  $\mathbf{O}^\perp: X = 0$ , we need only observe that the projection of the nonzero vector  $(X, AX) \in \mathbf{A}$  on  $\mathbf{O}^\perp$  is the vector  $(O, AX)$ , and

$$\frac{|(O, AX)|^2}{|(X, AX)|^2} = \frac{(AX)(AX)^*}{(1 + A^*A)XX^*} = \frac{AA^*}{1 + AA^*}$$

is independent of  $X$ .