

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	34 (1988)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	ISOCLINIC $n$ -PLANES IN $R^{2n}$ AND THE HOPF-STEENROD SPHERE BUNDLES $S^{2n-1} \rightarrow S^n$ , $n=2,4,8$
<b>Autor:</b>	Wong, Yung-Chow / Mok, Kam-Ping
<b>Anhang:</b>	Appendix 1. The Cayley numbers
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-56593">https://doi.org/10.5169/seals-56593</a>

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 08.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

correspond to the coordinate transformations  $t \rightarrow tB(\lambda)/N(\lambda)^{1/2}$  in  $\mathcal{I}_4$ , where  $B(\lambda)$  are the matrices given in (1.7) in Theorem 1.6. By Theorem 2.5, the elements  $B(\lambda)/N(\lambda)^{1/2}$  of  $SO(4)$  form a subgroup isomorphic with  $S^3$ . Therefore, the bundle group  $O(4)$  in  $\mathcal{HS}_4$  can be replaced by  $S^3$ . Similarly, the bundle group  $O(2)$  in  $\mathcal{HS}_2$  can be replaced by  $S^1$ . With these observations, we can now prove the following theorem by proceeding as in the proof of Theorem 5.3.

**THEOREM 5.4.** *The representative coordinate bundles constructed in § 4 for the sphere bundles  $\mathcal{HS}_2$  and  $\mathcal{HS}_4$ , with bundle groups  $S^1$  and  $S^3$  respectively, are topologically the same as the representative coordinate bundles constructed in § 3 for the sphere bundles  $\mathcal{I}_2$  and  $\mathcal{I}_4$ , respectively.*

Finally, we remark that representative coordinate bundles of the bundles  $\mathcal{SL}_n$  in Theorem 4.2 are topologically essentially the same as the representative coordinate bundles of the bundles  $\mathcal{IL}_n$  in Theorem 3.2.

#### APPENDIX 1. THE CAYLEY NUMBERS

The Cayley numbers, denoted by  $X, Y, Z, W$ , etc. are ordered pairs  $(q_1, q_2)$  of quaternions subject to the rules and having the properties listed below. The set of all Cayley numbers, therefore, forms a (non-commutative and non-associative) real division algebra. No proof of the properties will be given as they can all be checked by direct computations.

(1) The *addition* is defined by

$$(q_1, q_2) + (q'_1, q'_2) = (q_1 + q'_1, q_2 + q'_2).$$

The *zero* is  $O = (O, O)$ .

(2) The *multiplication* is defined by

$$(q_1, q_2)(q'_1, q'_2) = (q_1q'_1 - q'_2*q_2, q'_2q_1 + q_2q'_1*),$$

where  $q'_1*$ ,  $q'_2*$  are respectively the conjugates of (the quaternions)  $q'_1$ ,  $q'_2$ . The (two-sided) *unit* is  $1 \equiv (1, 0)$ .

(3) Multiplication is

(i) distributive with respect to addition, i.e.,

$$(X+Y)W = XW + YW, \quad W(X+Y) = WX + WY;$$

- (ii) not commutative, i.e., generally,  $XY \neq YX$  (but see (4) (iv) below);
  - (iii) not associative, i.e., generally,  $(XY)W \neq X(YW)$  (but see (7) below).
- (4) The *real part* of  $X \equiv (q_1, q_2)$  is  $\text{Re } X = (\text{Re } q_1, 0) \equiv \text{Re } q_1$ .  $X$  is said to be *real* if  $X = \text{Re } X$ ; i.e.,  $(q_1, q_2)$  is real iff  $q_1$  is real and  $q_2 = 0$ .
- (i)  $\text{Re}(X + Y) = \text{Re}(X) + \text{Re}(Y)$ .
  - (ii)  $\text{Re}(XY) = \text{Re}(YX)$ .
  - (iii)  $\text{Re}(CX) = 0$  for all  $X$  implies that  $C = 0$ .
  - (iv)  $CX = XC$  for all  $X$  iff  $C$  is real. In this case,  $C = (c_1, 0)$ , where  $c_1$  = real, and  $CX = (c_1 q_1, c_1 q_2) = XC$ .
- (5) The *conjugate* of  $X \equiv (q_1, q_2)$  is  $X^* = (q_1^*, -q_2)$ .
- (i)  $(X + Y)^* = X^* + Y^*$ ,
  - (ii)  $(XY)^* = Y^*X^*$ .
  - (iii)  $X^* = X$  iff  $X$  is real.
- (6) The *norm* of  $X$  is the non-negative real number  $N(X) \equiv XX^*$ , which is also equal to  $X^*X$ . The *length* of  $X$  is the non-negative real number  $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$ .
- (i)  $N(X) = 0$  iff  $X = 0$ .
  - (ii) If  $X \neq 0$ , then  $X^{-1} \equiv X^*/N(X)$  is a right and left inverse of  $X$ .
  - (iii)  $N(XY) = N(X)N(Y)$ . It follows from this that  $XY = 0$  iff  $X = 0$  or  $Y = 0$ .
- (7) Though multiplication is generally non-associative,
- (i)  $(XY)Y^* = X(YY^*)$ .
  - (ii) If  $Y \neq 0$ , then  $(XY)Y^{-1} = X = Y^{-1}(YX)$ .
  - (iii)  $\text{Re}((XY)W) = \text{Re}(X(YW))$ .

## APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of  $S^{2n-1}$  by  $S^{n-1}$  over  $S^n$ ,  $n = 2, 4$ , or  $8$ , by intersecting the unit sphere  $S^{2n-1}$  in  $R^{2n} = Q_n \times Q_n$  with the  $Q_n$ -lines  $Y = CX$  and  $X = 0$ . In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic  $n$ -planes in  $R^{2n}$  are equivalent concepts. Here we prove, directly, the