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correspond to the coordinate transformations  $t \rightarrow tB(\lambda)/N(\lambda)^{1/2}$  in  $\mathcal{I}_4$ , where  $B(\lambda)$  are the matrices given in (1.7) in Theorem 1.6. By Theorem 2.5, the elements  $B(\lambda)/N(\lambda)^{1/2}$  of  $SO(4)$  form a subgroup isomorphic with  $S^3$ . Therefore, the bundle group  $O(4)$  in  $\mathcal{H}\mathcal{S}_4$  can be replaced by  $S^3$ . Similarly, the bundle group  $O(2)$  in  $\mathcal{H}\mathcal{S}_2$  can be replaced by  $S^1$ . With these observations, we can now prove the following theorem by proceeding as in the proof of Theorem 5.3.

**THEOREM 5.4.** *The representative coordinate bundles constructed in § 4 for the sphere bundles  $\mathcal{H}\mathcal{S}_2$  and  $\mathcal{H}\mathcal{S}_4$ , with bundle groups  $S^1$  and  $S^3$  respectively, are topologically the same as the representative coordinate bundles constructed in § 3 for the sphere bundles  $\mathcal{I}_2$  and  $\mathcal{I}_4$ , respectively.*

Finally, we remark that representative coordinate bundles of the bundles  $\mathcal{S}\mathcal{L}_n$  in Theorem 4.2 are topologically essentially the same as the representative coordinate bundles of the bundles  $\mathcal{I}\mathcal{L}_n$  in Theorem 3.2.

#### APPENDIX 1. THE CAYLEY NUMBERS

The Cayley numbers, denoted by  $X, Y, Z, W$ , etc. are ordered pairs  $(q_1, q_2)$  of quaternions subject to the rules and having the properties listed below. The set of all Cayley numbers, therefore, forms a (non-commutative and non-associative) real division algebra. No proof of the properties will be given as they can all be checked by direct computations.

(1) The *addition* is defined by

$$(q_1, q_2) + (q'_1, q'_2) = (q_1 + q'_1, q_2 + q'_2).$$

The *zero* is  $O = (O, O)$ .

(2) The *multiplication* is defined by

$$(q_1, q_2)(q'_1, q'_2) = (q_1q'_1 - q_2^*q'_2, q'_2q_1 + q_2q_1^*),$$

where  $q_1^*, q_2^*$  are respectively the conjugates of (the quaternions)  $q_1, q_2$ . The (two-sided) *unit* is  $1 \equiv (1, 0)$ .

(3) Multiplication is

(i) distributive with respect to addition, i.e.,

$$(X + Y)W = XW + YW, \quad W(X + Y) = WX + WY;$$

- (ii) not commutative, i.e., generally,  $XY \neq YX$  (but see (4) (iv) below);
- (iii) not associative, i.e., generally,  $(XY)W \neq X(YW)$  (but see (7) below).
- (4) The *real part* of  $X \equiv (q_1, q_2)$  is  $\text{Re } X = (\text{Re } q_1, 0) \equiv \text{Re } q_1$ .  $X$  is said to be *real* if  $X = \text{Re } X$ ; i.e.,  $(q_1, q_2)$  is real iff  $q_1$  is real and  $q_2 = 0$ .
- (i)  $\text{Re}(X + Y) = \text{Re}(X) + \text{Re}(Y)$ .
- (ii)  $\text{Re}(XY) = \text{Re}(YX)$ .
- (iii)  $\text{Re}(CX) = 0$  for all  $X$  implies that  $C = 0$ .
- (iv)  $CX = XC$  for all  $X$  iff  $C$  is real. In this case,  $C = (c_1, 0)$ , where  $c_1 = \text{real}$ , and  $CX = (c_1q_1, c_1q_2) = XC$ .
- (5) The *conjugate* of  $X \equiv (q_1, q_2)$  is  $X^* = (q_1^*, -q_2)$ .
- (i)  $(X + Y)^* = X^* + Y^*$ ,
- (ii)  $(XY)^* = Y^*X^*$ .
- (iii)  $X^* = X$  iff  $X$  is real.
- (6) The *norm* of  $X$  is the non-negative real number  $N(X) \equiv XX^*$ , which is also equal to  $X^*X$ . The *length* of  $X$  is the non-negative real number  $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$ .
- (i)  $N(X) = 0$  iff  $X = 0$ .
- (ii) If  $X \neq 0$ , then  $X^{-1} \equiv X^*/N(X)$  is a right and left inverse of  $X$ .
- (iii)  $N(XY) = N(X)N(Y)$ . It follows from this that  $XY = 0$  iff  $X = 0$  or  $Y = 0$ .
- (7) Though multiplication is generally non-associative,
- (i)  $(XY)Y^* = X(Y Y^*)$ .
- (ii) If  $Y \neq 0$ , then  $(XY)Y^{-1} = X = Y^{-1}(YX)$ .
- (iii)  $\text{Re}((XY)W) = \text{Re}(X(YW))$ .

## APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of  $S^{2n-1}$  by  $S^{n-1}$  over  $S^n$ ,  $n = 2, 4$ , or  $8$ , by intersecting the unit sphere  $S^{2n-1}$  in  $R^{2n} = Q_n \times Q_n$  with the  $Q_n$ -lines  $Y = CX$  and  $X = 0$ . In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic  $n$ -planes in  $R^{2n}$  are equivalent concepts. Here we prove, directly, the