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Artikel: ISOCLINIC n-PLANES IN \$R^{2n}\$ AND THE HOPF-STEENROD

SPHERE BUNDLES \$\$^{2n-1} \rightarrow \$^n,\quad n=2,4,8\$

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**Kapitel:** 2. SOME FURTHER RESULTS

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Therefore,  $f(\Psi_n)$  is the set  $\Phi_n$  of mutually isoclinic *n*-planes in our Theorem 1.6.

## 2. Some further results

From now on we shall confine our attention to n-dimensional maximal sets of mutually isoclinic n-planes in  $R^{2n}$ , and therefore, n has always the values 2, 4, or 8 unless stated otherwise.

In this section, we prove a few more theorems for use in § 3. In these theorems, the indices a, b have the range of values (0, 1, ..., n-1);  $B_0 = I$  is the identity matrix of order n;  $B_1$ , ...,  $B_{n-1}$  are the  $n \times n$  matrices listed in Theorems 1.5 and 1.6;  $\lambda = (\lambda_a)$  is an ordered set of n real parameters; and

$$B(\lambda) \equiv \sum_{a} \lambda_{a} B_{a}$$
,  $N(\lambda) \equiv \sum_{a} \lambda_{a}^{2}$ .

Moreover, for any matrix M, we denote its transpose by  $M^T$ .

THEOREM 2.1.

- (i)  $B(\lambda)B(\lambda)^T = N(\lambda)I$ .
- (ii) If  $\lambda \neq 0$ , then

$$B(\lambda)^{-1} = B(\lambda)^T/N(\lambda) = \sum_a \lambda_a B_a^T/N(\lambda)$$
,

so that if  $\lambda \neq 0$ , the equation  $y = xB(\lambda)$  is equivalent to the equation  $x = yB(\mu)^T$ , where  $\mu = \lambda/N(\lambda) \neq 0$ .

(iii) 
$$\det B(\lambda) = + (N(\lambda))^{n/2}.$$

(iv) If  $N(\lambda) = 1$ , then  $B(\lambda) \in SO(n)$ , where SO(n) is the set of all orthogonal matrices of order n and determinant +1.

Proof. 
$$B(\lambda)B(\lambda)^{T} = \left(\sum_{a}\lambda_{a}B_{a}\right)\left(\sum_{b}\lambda_{b}B_{b}^{T}\right) = \sum_{a,b}\lambda_{a}\lambda_{b}B_{a}B_{b}^{T}$$
$$= \sum_{a}\lambda_{a}^{2}B_{a}B_{a}^{T} + \sum_{a < b}\lambda_{a}\lambda_{b}\left(B_{a}B_{b}^{T} + B_{b}B_{a}^{T}\right),$$

which, on account of the Hurwitz matrix equations (1.2), is equal to  $(\sum_a \lambda_a^2)I = N(\lambda)I$ . This proves (i), and also (ii). To prove (iii), we first note that since  $B(\lambda)$  is a square matrix of order n, det  $B(\lambda)$  is a homogeneous polynomial of degree n in the  $\lambda_a$ 's, and it follows from (i) that

$$(\det B(\lambda))^2 = \det (B(\lambda)B(\lambda)^T) = (N(\lambda))^n$$
.

Therefore,

(2.1) 
$$\det B(\lambda) = \pm (N(\lambda))^{n/2} = \pm (\lambda_0^2 + \lambda_1^2 + ... + \lambda_{n-1}^2)^{n/2}$$
$$= \pm (\lambda_0^n + \text{other product terms in } \lambda_a).$$

On the other hand, since  $B_0 = I$ , and  $B_1, ..., B_{n-1}$  are all skew-symmetric matrices, the diagonal elements of  $B(\lambda)$  are all equal to  $\lambda_0$ , and none of the other elements of  $B(\lambda)$  is equal to  $\lambda_0$ . Therefore,

$$\det B(\lambda) = \lambda_0^n + \text{other product terms in } \lambda_a$$
.

Comparison of this with (2.1) gives (iii). Finally, (iv) follows immediately from (i) and (iii).

Returning to Theorems 1.2 and 1.6, we now prove

THEOREM 2.2. Let  $\Phi_n$  be the maximal set of mutually isoclinic n-planes in  $R^{2n}$  described in Theorem 1.6, and let (u, v) be any vector in  $R^{2n}$ . If  $u \neq 0$ , then the unique n-plane in  $\Phi_n$  containing (u, v) is

$$(2.2) y = x [vu^T - (vB_1u^T)B_1 - ... - (vB_{n-1}u^T)B_{n-1}]/(uu)^T.$$

If  $v \neq 0$ , then the unique n-plane in  $\Phi_n$  containing (u, v) is

$$(2.3) x = y \left[ uv^T - (uB_1^T v^T) B_1^T - \dots - (uB_{n-1}^T v^T) B_{n-1}^T \right] / (vv)^T.$$

Here,  $B_1$ , ...,  $B_{n-1}$  are the matrices in (1.3), (1.4), or (1.5) according as n = 2, 4, or 8.

*Proof.* We shall prove only (2.2) for the case  $u \neq 0$ , as (2.3) for the case  $v \neq 0$  can be proved similarly. Suppose that  $u \neq 0$  and

$$(2.4) y = x(\lambda_0 + \lambda_1 B_1 + ... + \lambda_{n-1} B_{n-1})$$

is an *n*-plane in  $\Phi_n$  containing (u, v). Then we have

$$v = u(\lambda_0 + \lambda_1 B_1 + ... + \lambda_{n-1} B_{n-1}),$$

which can be written as

$$v = [\lambda_0 \lambda_1 \dots \lambda_{n-1}] \begin{bmatrix} u \\ uB_1 \\ \vdots \\ uB_{n-1} \end{bmatrix}.$$

Multiplying the two sides of this equation on the right by

$$[u^T, -B_1u^T, ..., -B_{n-1}u^T]$$

and making use of the Hurwitz matrix equations (1.2), we get

$$v[u^T, -B_1u^T, ..., -B_{n-1}u^T] = [\lambda_0\lambda_1 ... \lambda_{n-1}] (uu^T)I.$$

Since  $uu^T \neq 0$ , the above equation determines the  $\lambda_a$ 's uniquely in terms of u, v. Now with these values of  $\lambda_a$ 's, equation (2.4) becomes equation (2.2), as we wanted to prove. Incidentally, the above proof also confirms that there is exactly one n-plane in  $\Phi_n$  containing the vector (u, v) (cf. Theorem 1.2).

Next, we give a direct proof of Theorem 1.3 for the special cases n = 2, 4, 8, and state the result as

THEOREM 2.3. The maximal set  $\Phi_n = \{x = 0, y = xB(\lambda)\}$  of mutually isoclinic n-planes in  $R^{2n}$ , n = 2, 4, or 8, can be given a differentiable structure so that it is diffeomorphic with the n-sphere  $S^n$ .

Proof. Let us regard  $\Phi_n$  as a point set whose elements are the *n*-planes in  $\Phi_n$ . Then, the subset  $\Phi_n \backslash \mathbf{O}^\perp = \{y = xB(\lambda)\}$  of  $\Phi_n$  is an open subset in which we can define a coordinate system by assigning to the element  $y = xB(\lambda)$  the coordinate  $\lambda = (\lambda_0, \lambda_1, ..., \lambda_{n-1})$ . The subset  $\Phi_n \backslash \mathbf{O} = \{x = 0 \text{ and } y = xB(\lambda), \text{ where } \lambda \neq 0\}$  of  $\Phi_n$  is also an open subset. By Theorem 2.1 (ii), this subset is the same as the subset  $\{x = yB(\mu)^T\}$ , and so, we can define in it a coordinate system by assigning to the element  $x = yB(\mu)^T$  the coordinate  $\mu = (\mu_0, \mu_1, ..., \mu_{n-1})$ . Thus  $\Phi_n$  is covered by the two coordinate neighborhoods

(2.5) 
$$(\Phi_n \backslash \mathbf{O}^{\perp}, \lambda), \quad (\Phi_n \backslash \mathbf{O}, \mu).$$

Moreover, we can see from Theorem 2.1 (ii) that for any element in  $(\Phi_n \backslash \mathbf{O}^{\perp}) \cap (\Phi_n \backslash \mathbf{O}) = \Phi_n \backslash \{\mathbf{O}^{\perp}, \mathbf{O}\}$ , its two coordinates  $\lambda$ ,  $\mu$ , both nonzero, are related by

(2.6) 
$$\mu = \lambda/N(\lambda)$$
, or equivalently,  $\lambda = \mu/N(\mu)$ .

Hence,  $\Phi_n$  is an *n*-dimensional manifold.

To show that  $\Phi_n$  is diffeomorphic with the *n*-sphere  $S^n$ , we view  $S^n$  as the unit sphere  $x_1^2 + ... + x_{n+1}^2 = 1$  in  $R^{n+1}$ , and use stereographic projections. Let  $q_1(0, ..., 0, 1)$  and  $q_2(0, ..., 0, -1)$  be respectively the north and south poles of  $S^n$ . Then  $S^n$  is the union of the two open subsets

 $S^n \setminus q_1$  and  $S^n \setminus q_2$ . For an arbitrary point q in  $S^n \setminus q_1$ , let the line  $q_1 q$  meet the equator n-plane  $x_{n+1} = 0$  at the point  $(\lambda, 0)$ ; and for an arbitrary point q in  $S^n \setminus q_2$ , let the line  $q_2 q$  meet the equator n-plane  $x_{n+1} = 0$  at the point  $(\mu, 0)$ . Then  $S^n$  is covered by the two coordinate neighborhoods

$$(S^{n}\backslash q_{1},\lambda), \quad (S^{n}\backslash q_{2},\mu).$$

Moreover, it is easy to verify that for a point in  $S^n \setminus \{q_1, q_2\}$ , its two coordinates  $\lambda$  and  $\mu$  are also both nonzero and related by (2.6).

It now follows from (2.5), (2.6) and (2.7) that if  $f_1$  is the map from  $\Phi_n \backslash \mathbf{O}^{\perp}$  to  $S^n \backslash q_1$  sending an *n*-plane in  $\Phi_n \backslash \mathbf{O}^{\perp}$  with coordinate  $\lambda$  to the point in  $S^n \backslash q_1$  with the same coordinate  $\lambda$ , and  $f_2$  is the map from  $\Phi_n \backslash \mathbf{O}$  to  $S^n \backslash q_2$  sending an *n*-plane in  $\Phi_n \backslash \mathbf{O}$  with coordinate  $\mu$  to the point in  $S^n \backslash q_2$  with the same coordinate  $\mu$ , then  $f_1$ ,  $f_2$  combined will give a diffeomorphism from  $\Phi_n$  to  $S^n$ .

In the remainder of this section, we are concerned exclusively with the matrices  $B(\lambda)$  with  $N(\lambda) = 1$ . For convenience, we shall denote such matrices by  $B(\lambda')$ , with the understanding that  $\lambda'$  always satisfies the condition  $N(\lambda') = 1$ .

We know from Theorem 2.1 (iv) that every  $B(\lambda')$  belongs to SO(n). Let us now regard SO(n) as the special orthogonal group. Then the set of elements  $B(\lambda')$  of SO(n) will generate a subgroup of SO(n). We wish to know what this subgroup of SO(n) is, and the next three theorems will give us the answer.

THEOREM 2.4. For n = 2, the set of elements  $B(\lambda')$  forms the group SO(2) which is isomorphic with  $S^1$ .

Proof. Since

$$B(\lambda') = \begin{bmatrix} \lambda'_0 & \lambda'_1 \\ -\lambda'_1 & \lambda'_0 \end{bmatrix} \quad \text{and} \quad \det B(\lambda') = (\lambda'_0)^2 + (\lambda'_1)^2 = 1 ,$$

the elements of SO(2) are the elements  $B(\lambda')$  themselves.

Theorem 2.5. For n = 4, the set of elements  $B(\lambda')$  forms a 3-parameter subgroup of SO(4), isomorphic with  $S^3$ .

*Proof.* First, since  $N(\lambda') = (\lambda'_0)^2 + ... + (\lambda'_3)^2 = 1$ , the set  $B(\lambda')$ , with a natural topology, is homeomorphic with the unit 3-sphere  $S^3$  in  $R^4$ . Next, using (1.4), we can easily verify that

$$B_2B_3 = -B_1$$
,  $B_3B_1 = -B_2$ ,  $B_1B_2 = -B_3$ .

With this and Theorem 2.1 (ii), straight forward computation will show that for any two elements  $B(\lambda')$  and  $B(\mu')$  of SO(4), the product  $B(\lambda')B(\mu')^{-1}$  is an element of SO(4) of the form  $B(\nu')$ , where the components of  $\nu'$  are analytic functions of the components of  $\lambda'$  and  $\mu'$ . This proves our theorem.

For the case n=8, we first observe that the elements  $B(\lambda')$  of SO(8) do not, by themselves, form a subgroup of SO(8). For example, although  $B_1$ ,  $B_2$  are both of the form  $B(\lambda')$ , their product  $B_1B_2$  is not. In fact, we have

Theorem 2.6. For n = 8, the set of elements  $B(\lambda')$  of SO(8) generates the group SO(8) itself.

*Proof.* Our proof consists of two steps (i) and (ii). In (i), we prove that the 28 skew-symmetric  $8 \times 8$  matrices  $B_i$ ,  $B_iB_j$  (i, j=1,...,7, and i < j) are linearly independent. In (ii), we prove that the Lie algebra of the subgroup of SO(8) generated by the elements  $B(\lambda')$  coincides with the Lie algebra o(8) of SO(8). The assertion in our theorem then follows from the well-known fact in Lie groups that there is a one-one correspondence between the connected Lie subgroups of a Lie group G and the Lie subalgebras of the Lie algebra of G.

(i) From (1.5), we see that the  $8 \times 8$  matrices  $B_i (i = 1, ..., 7)$  can be partitioned as

$$B_6 = \left[ egin{array}{cccc} & & I & & I \ & & K & & \ & -K & & & \ \end{array} 
ight], \quad B_7 = \left[ egin{array}{cccc} & & J \ & L \ & -L & \ \end{array} 
ight] \; ,$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
,  $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,

are  $2 \times 2$  submatrices and each empty space represents a  $2 \times 2$  zero-matrix 0. Since the matrices I, J, K, L have the properties

$$I^2 = I$$
,  $J^2 = -I$ ,  $K^2 = I$ ,  $L^2 = I$ ,  
 $JK = -KJ = -L$ ,  $KL = -LK = J$ ,  $LJ = -JL = -K$ ,

we can easily verify that the products  $B_i B_j (i, j = 1, ..., 7, \text{ and } i < j)$  are matrices of the same form as  $B_i$ , having some of O,  $\pm I$ ,  $\pm J$ ,  $\pm K$ ,  $\pm L$  as  $2 \times 2$  submatrices.

To prove that the 28 matrices  $B_i$ ,  $B_iB_j$  are linearly independent, we construct the  $8 \times 8$  matrix

$$M \equiv \sum_{i} \alpha_{i} B_{i} + \sum_{i < j} \alpha_{ij} (B_{i} B_{j}),$$

where the  $\alpha$ 's are some real numbers, and show that if M=0, then all the  $\alpha$ 's are zero. Let  $M=[M_{hk}]$ , where  $M_{hk}(h, k=1, 2, 3, 4)$  are the  $2\times 2$  submatrices of M. Then by using the explicit forms of  $B_i$  and  $B_iB_j$ , we can write M as the sum of the following four matrices:

$$\begin{bmatrix} M_{11} & & & & & \\ & M_{22} & & & & \\ & & M_{33} & & \\ & & & M_{44} \end{bmatrix} = \alpha_1 \begin{bmatrix} J & & & & \\ & J & & & \\ & & J & & \\ & & & -J \end{bmatrix} + \alpha_{23} \begin{bmatrix} -J & & & \\ & -J & & \\ & & J & \\ & & -J \end{bmatrix}$$

$$+ lpha_{45} egin{bmatrix} -J & & & & \ & J & & & \ & & -J & & \ \end{pmatrix} + lpha_{67} egin{bmatrix} J & & & & \ & -J & & \ & & -J & \ & & -J & \ \end{pmatrix} ,$$

$$\begin{bmatrix} M_{12} & M_{12} & & & \\ M_{21} & M_{34} & \end{bmatrix} = \alpha_{2} \begin{bmatrix} -K & & & \\ -K & & & I \end{bmatrix} + \alpha_{13} \begin{bmatrix} -K & & & \\ -K & & & -I \end{bmatrix} + \alpha_{14} \begin{bmatrix} -L & & & \\ & & & & I \end{bmatrix} + \alpha_{12} \begin{bmatrix} L & -L & & \\ & & & & I \end{bmatrix} + \alpha_{44} \begin{bmatrix} -I & & & \\ & & & & -K \end{bmatrix} + \alpha_{57} \begin{bmatrix} I & & & \\ & & & & -K \end{bmatrix} + \alpha_{47} \begin{bmatrix} -I & & & \\ -J & & & & \\ & & & & -K \end{bmatrix} + \alpha_{56} \begin{bmatrix} J & & & \\ & & & & -K \end{bmatrix} + \alpha_{15} \begin{bmatrix} & & & & \\ -K & & & I \end{bmatrix} + \alpha_{15} \begin{bmatrix} & & & & \\ -K & & & I \end{bmatrix} + \alpha_{15} \begin{bmatrix} & & & & \\ -K & & & & \\ -I & & & & \end{bmatrix} + \alpha_{14} \begin{bmatrix} & & & & \\ & & & -J \end{bmatrix} + \alpha_{14} \begin{bmatrix} & & & & \\ & & & -J \end{bmatrix} + \alpha_{26} \begin{bmatrix} & & & & \\ -I & & & & \\ & & & & & \end{bmatrix} + \alpha_{37} \begin{bmatrix} & & & & & \\ -I & & & & \\ & & & & & \end{bmatrix} + \alpha_{27} \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ -J & & & & \\ & & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ -J & & & & \\ & & & & & \\ \end{bmatrix} + \alpha_{27} \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ -J & & & & \\ \end{bmatrix} + \alpha_{26} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ -J & & & & \\ \end{bmatrix} + \alpha_{27} \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha_{36} \begin{bmatrix} & & & & & \\ & & & & \\ \end{bmatrix} + \alpha$$

$$\begin{bmatrix} M_{14} \\ M_{32} \end{bmatrix} = \alpha_{6} \begin{bmatrix} I \\ -K \end{bmatrix} + \alpha_{17} \begin{bmatrix} I \\ -K \end{bmatrix} + \alpha_{17} \begin{bmatrix} I \\ -K \end{bmatrix}$$

$$+ \alpha_{7} \begin{bmatrix} I \\ J \end{bmatrix} + \alpha_{16} \begin{bmatrix} I \\ J \end{bmatrix} + \alpha_{16} \begin{bmatrix} I \\ J \end{bmatrix} + \alpha_{16} \begin{bmatrix} I \\ J \end{bmatrix}$$

$$+ \alpha_{24} \begin{bmatrix} I \\ K \end{bmatrix} + \alpha_{35} \begin{bmatrix} I \\ -K \end{bmatrix} + \alpha_{35} \begin{bmatrix} I \\ -K \end{bmatrix}$$

$$+ \alpha_{25} \begin{bmatrix} I \\ -J \end{bmatrix} + \alpha_{34} \begin{bmatrix} I \\ J \end{bmatrix} + \alpha_{34} \begin{bmatrix} I \\ J \end{bmatrix}$$

Now, M=0 means that all its submatrices  $M_{hk}$  are zero. Since I, J, K, L are linearly independent, the equations  $M_{hk}=0$  are equivalent to a number of linear equations in the  $\alpha$ 's, and from these linear equations we can easily see that the  $\alpha$ 's must all be zero. For example, it is obvious from the equations

$$M_{12} = (\alpha_2 + \alpha_{13})K + (\alpha_3 - \alpha_{12})L - (\alpha_{46} + \alpha_{57})I - (\alpha_{47} - \alpha_{56})J = 0,$$
  

$$M_{34} = (\alpha_2 - \alpha_{13})I + (\alpha_3 + \alpha_{12})J + (-\alpha_{46} + \alpha_{57})K - (\alpha_{47} + \alpha_{56})L = 0$$

that

$$\alpha_2$$
,  $\alpha_{13}$ ,  $\alpha_3$ ,  $\alpha_{12}$ ,  $\alpha_{46}$ ,  $\alpha_{57}$ ,  $\alpha_{47}$ ,  $\alpha_{56}$ 

must all be zero. Thus we have proved that the 28 matrices  $B_i$ ,  $B_iB_j$  are linearly independent.

(ii) Let G be the Lie subgroup of SO(8) generated by the elements  $B(\lambda')$ , and g its Lie algebra. Then g is a Lie subalgebra of the Lie algebra o(8) of SO(8). We now prove that in fact g = o(8).

From the theory of Lie groups we know that if  $t \to f(t)$ , where  $t \in R$  and  $f(t) \in G$ , is any curve in G passing through the identity element

I = f(0) of G, then the velocity vector f'(0) of this curve at I is an element of g. Now

$$t \rightarrow f_i(t) \equiv (\cos t)I + (\sin t)B_i \quad (i=1, ..., 7)$$

are obviously curves in G such that  $f_i(0) = I$  and  $f'_i(0) = B_i$ . Therefore,  $B_i$  are all elements of g.

Since g is a Lie subalgebra of o(8) and  $B_i \in g$ , the Lie products  $[B_i, B_j] = B_i B_j - B_j B_i = 2B_i B_j$ , where i, j = 1, ..., 7, and i < j, are all in g.

We have thus proved that the 28 linearly independent skew-symmetric matrices,  $B_i$ ,  $B_iB_j$  all belong to  $g \subset o(8)$ . Since o(8) is the Lie algebra of all skew-symmetric matrices of order 8 and is therefore of dimension 28, g coincides with o(8). This completes the proof of Theorem 2.6.

3. The sphere bundles  $S^{2n-1} \to \Phi_n$ , n=2,4, or 8, with fibers on mutually isoclinic n-planes in  $R^{2n}$ 

In  $R^{2n}$ , n=2,4, or 8, provided with rectangular coordinate system (x,y), let  $S^{2n-1}$  be the unit sphere and  $\Phi_n$  the maximal set of mutually isoclinic n-planes  $\{x=0, y=xB(\lambda)\}$  defined in Theorem 1.6. Then with the preparations we have made in § 2, we can now prove

Theorem 3.1. In  $R^{2n}$ , n=2,4, or 8, the n-planes in the maximal set  $\Phi_n$  of mutually isoclinic n-planes slice the unit sphere  $S^{2n-1}$  into a fiber bundle

$$\mathcal{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n),$$

with base space  $\Phi_n$ , projection  $\pi$ , fiber  $S^{n-1}$  and group  $G_n$ , where  $G_2 = S^1$ ,  $G_4 = S^3$ , and  $G_8 = SO(8)$ .

*Proof.* We prove by exhibiting all the ingredients of a representative coordinate bundle.

- (1) The bundle space  $S^{2n-1}$  has the equation  $xx^T + yy^T = 1$  in  $R^{2n}$ .
- (2) The base space  $\Phi_n$  is covered by the two coordinate systems

(2.5) 
$$(\Phi_n \backslash \mathbf{O}^{\perp}, \lambda), \quad (\Phi_n \backslash \mathbf{O}, \mu)$$

as in the proof of Theorem 2.3, where  $\mathbf{O}^{\perp}$  is the *n*-plane x=0,  $\mathbf{O}$  is the *n*-plane y=0,  $\lambda$  is the parameter in the equation  $y=xB(\lambda)$  of an *n*-plane in  $\Phi_n\backslash\mathbf{O}^{\perp}$ , and  $\mu$  is the parameter in the equation  $x=yB(\mu)^T$  of