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**Artikel:** ISOCLINIC  $n$ -PLANES IN  $\mathbb{R}^{2n}$  AND THE HOPF-STEENROD SPHERE BUNDLES  $S^{2n-1} \rightarrow S^n, \text{quad } n=2,4,8$   
**Autor:** Wong, Yung-Chow / Mok, Kam-Ping  
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Therefore,  $f(\Psi_n)$  is the set  $\Phi_n$  of mutually isoclinic  $n$ -planes in our Theorem 1.6.

## 2. SOME FURTHER RESULTS

From now on we shall confine our attention to  $n$ -dimensional maximal sets of mutually isoclinic  $n$ -planes in  $R^{2n}$ , and therefore,  $n$  has always the values 2, 4, or 8 unless stated otherwise.

In this section, we prove a few more theorems for use in § 3. In these theorems, the indices  $a, b$  have the range of values  $(0, 1, \dots, n-1)$ ;  $B_0 = I$  is the identity matrix of order  $n$ ;  $B_1, \dots, B_{n-1}$  are the  $n \times n$  matrices listed in Theorems 1.5 and 1.6;  $\lambda = (\lambda_a)$  is an ordered set of  $n$  real parameters; and

$$B(\lambda) \equiv \sum_a \lambda_a B_a, \quad N(\lambda) \equiv \sum_a \lambda_a^2.$$

Moreover, for any matrix  $M$ , we denote its transpose by  $M^T$ .

### THEOREM 2.1.

- (i)  $B(\lambda)B(\lambda)^T = N(\lambda)I$ .
- (ii) If  $\lambda \neq 0$ , then

$$B(\lambda)^{-1} = B(\lambda)^T/N(\lambda) = \sum_a \lambda_a B_a^T/N(\lambda),$$

so that if  $\lambda \neq 0$ , the equation  $y = xB(\lambda)$  is equivalent to the equation  $x = yB(\mu)^T$ , where  $\mu = \lambda/N(\lambda) \neq 0$ .

$$(iii) \quad \det B(\lambda) = + (N(\lambda))^{n/2}.$$

(iv) If  $N(\lambda) = 1$ , then  $B(\lambda) \in SO(n)$ , where  $SO(n)$  is the set of all orthogonal matrices of order  $n$  and determinant  $+1$ .

$$\begin{aligned} \text{Proof.} \quad B(\lambda)B(\lambda)^T &= (\sum_a \lambda_a B_a) (\sum_b \lambda_b B_b^T) = \sum_{a,b} \lambda_a \lambda_b B_a B_b^T \\ &= \sum_a \lambda_a^2 B_a B_a^T + \sum_{a < b} \lambda_a \lambda_b (B_a B_b^T + B_b B_a^T), \end{aligned}$$

which, on account of the Hurwitz matrix equations (1.2), is equal to  $(\sum_a \lambda_a^2)I = N(\lambda)I$ . This proves (i), and also (ii). To prove (iii), we first note that since  $B(\lambda)$  is a square matrix of order  $n$ ,  $\det B(\lambda)$  is a homogeneous polynomial of degree  $n$  in the  $\lambda_a$ 's, and it follows from (i) that

$$(\det B(\lambda))^2 = \det (B(\lambda)B(\lambda)^T) = (N(\lambda))^n.$$

Therefore,

$$(2.1) \quad \det B(\lambda) = \pm (N(\lambda))^{n/2} = \pm (\lambda_0^2 + \lambda_1^2 + \dots + \lambda_{n-1}^2)^{n/2} \\ = \pm (\lambda_0^n + \text{other product terms in } \lambda_a).$$

On the other hand, since  $B_0 = I$ , and  $B_1, \dots, B_{n-1}$  are all skew-symmetric matrices, the diagonal elements of  $B(\lambda)$  are all equal to  $\lambda_0$ , and none of the other elements of  $B(\lambda)$  is equal to  $\lambda_0$ . Therefore,

$$\det B(\lambda) = \lambda_0^n + \text{other product terms in } \lambda_a.$$

Comparison of this with (2.1) gives (iii). Finally, (iv) follows immediately from (i) and (iii).

Returning to Theorems 1.2 and 1.6, we now prove

**THEOREM 2.2.** *Let  $\Phi_n$  be the maximal set of mutually isoclinic  $n$ -planes in  $R^{2n}$  described in Theorem 1.6, and let  $(u, v)$  be any vector in  $R^{2n}$ . If  $u \neq 0$ , then the unique  $n$ -plane in  $\Phi_n$  containing  $(u, v)$  is*

$$(2.2) \quad y = x[vu^T - (vB_1u^T)B_1 - \dots - (vB_{n-1}u^T)B_{n-1}]/(uu)^T.$$

*If  $v \neq 0$ , then the unique  $n$ -plane in  $\Phi_n$  containing  $(u, v)$  is*

$$(2.3) \quad x = y[uv^T - (uB_1^T v^T)B_1^T - \dots - (uB_{n-1}^T v^T)B_{n-1}^T]/(vv)^T.$$

*Here,  $B_1, \dots, B_{n-1}$  are the matrices in (1.3), (1.4), or (1.5) according as  $n = 2, 4$ , or  $8$ .*

*Proof.* We shall prove only (2.2) for the case  $u \neq 0$ , as (2.3) for the case  $v \neq 0$  can be proved similarly. Suppose that  $u \neq 0$  and

$$(2.4) \quad y = x(\lambda_0 + \lambda_1 B_1 + \dots + \lambda_{n-1} B_{n-1})$$

is an  $n$ -plane in  $\Phi_n$  containing  $(u, v)$ . Then we have

$$v = u(\lambda_0 + \lambda_1 B_1 + \dots + \lambda_{n-1} B_{n-1}),$$

which can be written as

$$v = [\lambda_0 \lambda_1 \dots \lambda_{n-1}] \begin{bmatrix} u \\ uB_1 \\ \vdots \\ uB_{n-1} \end{bmatrix}.$$

Multiplying the two sides of this equation on the right by

$$[u^T, -B_1 u^T, \dots, -B_{n-1} u^T]$$

and making use of the Hurwitz matrix equations (1.2), we get

$$v[u^T, -B_1 u^T, \dots, -B_{n-1} u^T] = [\lambda_0 \lambda_1 \dots \lambda_{n-1}] (uu^T)I.$$

Since  $uu^T \neq 0$ , the above equation determines the  $\lambda_a$ 's uniquely in terms of  $u, v$ . Now with these values of  $\lambda_a$ 's, equation (2.4) becomes equation (2.2), as we wanted to prove. Incidentally, the above proof also confirms that there is exactly one  $n$ -plane in  $\Phi_n$  containing the vector  $(u, v)$  (cf. Theorem 1.2).

Next, we give a direct proof of Theorem 1.3 for the special cases  $n = 2, 4, 8$ , and state the result as

**THEOREM 2.3.** *The maximal set  $\Phi_n = \{x = 0, y = xB(\lambda)\}$  of mutually isoclinic  $n$ -planes in  $R^{2n}$ ,  $n = 2, 4$ , or  $8$ , can be given a differentiable structure so that it is diffeomorphic with the  $n$ -sphere  $S^n$ .*

*Proof.* Let us regard  $\Phi_n$  as a point set whose elements are the  $n$ -planes in  $\Phi_n$ . Then, the subset  $\Phi_n \setminus \mathbf{O}^\perp = \{y = xB(\lambda)\}$  of  $\Phi_n$  is an open subset in which we can define a coordinate system by assigning to the element  $y = xB(\lambda)$  the coordinate  $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n-1})$ . The subset  $\Phi_n \setminus \mathbf{O} = \{x = 0 \text{ and } y = xB(\lambda), \text{ where } \lambda \neq 0\}$  of  $\Phi_n$  is also an open subset. By Theorem 2.1 (ii), this subset is the same as the subset  $\{x = yB(\mu)^T\}$ , and so, we can define in it a coordinate system by assigning to the element  $x = yB(\mu)^T$  the coordinate  $\mu = (\mu_0, \mu_1, \dots, \mu_{n-1})$ . Thus  $\Phi_n$  is covered by the two coordinate neighborhoods

$$(2.5) \quad (\Phi_n \setminus \mathbf{O}^\perp, \lambda), \quad (\Phi_n \setminus \mathbf{O}, \mu).$$

Moreover, we can see from Theorem 2.1 (ii) that for any element in  $(\Phi_n \setminus \mathbf{O}^\perp) \cap (\Phi_n \setminus \mathbf{O}) = \Phi_n \setminus \{\mathbf{O}^\perp, \mathbf{O}\}$ , its two coordinates  $\lambda, \mu$ , both nonzero, are related by

$$(2.6) \quad \mu = \lambda/N(\lambda), \text{ or equivalently, } \lambda = \mu/N(\mu).$$

Hence,  $\Phi_n$  is an  $n$ -dimensional manifold.

To show that  $\Phi_n$  is diffeomorphic with the  $n$ -sphere  $S^n$ , we view  $S^n$  as the unit sphere  $x_1^2 + \dots + x_{n+1}^2 = 1$  in  $R^{n+1}$ , and use stereographic projections. Let  $q_1(0, \dots, 0, 1)$  and  $q_2(0, \dots, 0, -1)$  be respectively the north and south poles of  $S^n$ . Then  $S^n$  is the union of the two open subsets

$S^n \setminus q_1$  and  $S^n \setminus q_2$ . For an arbitrary point  $q$  in  $S^n \setminus q_1$ , let the line  $q_1 q$  meet the equator  $n$ -plane  $x_{n+1} = 0$  at the point  $(\lambda, 0)$ ; and for an arbitrary point  $q$  in  $S^n \setminus q_2$ , let the line  $q_2 q$  meet the equator  $n$ -plane  $x_{n+1} = 0$  at the point  $(\mu, 0)$ . Then  $S^n$  is covered by the two coordinate neighborhoods

$$(2.7) \quad (S^n \setminus q_1, \lambda), \quad (S^n \setminus q_2, \mu).$$

Moreover, it is easy to verify that for a point in  $S^n \setminus \{q_1, q_2\}$ , its two coordinates  $\lambda$  and  $\mu$  are also both nonzero and related by (2.6).

It now follows from (2.5), (2.6) and (2.7) that if  $f_1$  is the map from  $\Phi_n \setminus \mathbf{O}^\perp$  to  $S^n \setminus q_1$  sending an  $n$ -plane in  $\Phi_n \setminus \mathbf{O}^\perp$  with coordinate  $\lambda$  to the point in  $S^n \setminus q_1$  with the same coordinate  $\lambda$ , and  $f_2$  is the map from  $\Phi_n \setminus \mathbf{O}$  to  $S^n \setminus q_2$  sending an  $n$ -plane in  $\Phi_n \setminus \mathbf{O}$  with coordinate  $\mu$  to the point in  $S^n \setminus q_2$  with the same coordinate  $\mu$ , then  $f_1, f_2$  combined will give a diffeomorphism from  $\Phi_n$  to  $S^n$ .

In the remainder of this section, we are concerned exclusively with the matrices  $B(\lambda)$  with  $N(\lambda) = 1$ . For convenience, we shall denote such matrices by  $B(\lambda')$ , with the understanding that  $\lambda'$  always satisfies the condition  $N(\lambda') = 1$ .

We know from Theorem 2.1 (iv) that every  $B(\lambda')$  belongs to  $SO(n)$ . Let us now regard  $SO(n)$  as the special orthogonal group. Then the set of elements  $B(\lambda')$  of  $SO(n)$  will generate a subgroup of  $SO(n)$ . We wish to know what this subgroup of  $SO(n)$  is, and the next three theorems will give us the answer.

**THEOREM 2.4.** *For  $n = 2$ , the set of elements  $B(\lambda')$  forms the group  $SO(2)$  which is isomorphic with  $S^1$ .*

*Proof.* Since

$$B(\lambda') = \begin{bmatrix} \lambda'_0 & \lambda'_1 \\ -\lambda'_1 & \lambda'_0 \end{bmatrix} \quad \text{and} \quad \det B(\lambda') = (\lambda'_0)^2 + (\lambda'_1)^2 = 1,$$

the elements of  $SO(2)$  are the elements  $B(\lambda')$  themselves.

**THEOREM 2.5.** *For  $n = 4$ , the set of elements  $B(\lambda')$  forms a 3-parameter subgroup of  $SO(4)$ , isomorphic with  $S^3$ .*

*Proof.* First, since  $N(\lambda') = (\lambda'_0)^2 + \dots + (\lambda'_3)^2 = 1$ , the set  $B(\lambda')$ , with a natural topology, is homeomorphic with the unit 3-sphere  $S^3$  in  $R^4$ . Next, using (1.4), we can easily verify that

$$B_2 B_3 = -B_1, \quad B_3 B_1 = -B_2, \quad B_1 B_2 = -B_3.$$

With this and Theorem 2.1 (ii), straight forward computation will show that for any two elements  $B(\lambda')$  and  $B(\mu')$  of  $SO(4)$ , the product  $B(\lambda')B(\mu')^{-1}$  is an element of  $SO(4)$  of the form  $B(v')$ , where the components of  $v'$  are analytic functions of the components of  $\lambda'$  and  $\mu'$ . This proves our theorem.

For the case  $n = 8$ , we first observe that the elements  $B(\lambda')$  of  $SO(8)$  do not, by themselves, form a subgroup of  $SO(8)$ . For example, although  $B_1, B_2$  are both of the form  $B(\lambda')$ , their product  $B_1 B_2$  is not. In fact, we have

**THEOREM 2.6.** *For  $n = 8$ , the set of elements  $B(\lambda')$  of  $SO(8)$  generates the group  $SO(8)$  itself.*

*Proof.* Our proof consists of two steps (i) and (ii). In (i), we prove that the 28 skew-symmetric  $8 \times 8$  matrices  $B_i, B_i B_j (i, j = 1, \dots, 7, \text{ and } i < j)$  are linearly independent. In (ii), we prove that the Lie algebra of the subgroup of  $SO(8)$  generated by the elements  $B(\lambda')$  coincides with the Lie algebra  $\mathfrak{o}(8)$  of  $SO(8)$ . The assertion in our theorem then follows from the well-known fact in Lie groups that there is a one-one correspondence between the connected Lie subgroups of a Lie group  $G$  and the Lie subalgebras of the Lie algebra of  $G$ .

(i) From (1.5), we see that the  $8 \times 8$  matrices  $B_i (i = 1, \dots, 7)$  can be partitioned as

$$B_1 = \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix}, \quad B_2 = \begin{bmatrix} & K & & \\ -K & & & \\ & & I & \\ & & & -I \end{bmatrix}, \quad B_3 = \begin{bmatrix} & & L & \\ & -L & & \\ & & & J \\ & & J & \end{bmatrix}$$

$$B_4 = \begin{bmatrix} & K & & \\ & & -I & \\ -K & & & \\ & I & & \end{bmatrix}, \quad B_5 = \begin{bmatrix} & & L & \\ & & & -J \\ -L & & & \\ & -J & & \end{bmatrix},$$

$$B_6 = \begin{bmatrix} & & I \\ & K & \\ -I & -K & \end{bmatrix}, \quad B_7 = \begin{bmatrix} & & J \\ & L & \\ J & -L & \end{bmatrix},$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

are  $2 \times 2$  submatrices and each empty space represents a  $2 \times 2$  zero-matrix 0.

Since the matrices  $I, J, K, L$  have the properties

$$I^2 = I, \quad J^2 = -I, \quad K^2 = I, \quad L^2 = I, \\ JK = -KJ = -L, \quad KL = -LK = J, \quad LJ = -JL = -K,$$

we can easily verify that the products  $B_i B_j (i, j = 1, \dots, 7, \text{ and } i < j)$  are matrices of the same form as  $B_i$ , having some of  $O, \pm I, \pm J, \pm K, \pm L$  as  $2 \times 2$  submatrices.

To prove that the 28 matrices  $B_i, B_i B_j$  are linearly independent, we construct the  $8 \times 8$  matrix

$$M \equiv \sum_i \alpha_i B_i + \sum_{i < j} \alpha_{ij} (B_i B_j),$$

where the  $\alpha$ 's are some real numbers, and show that if  $M = 0$ , then all the  $\alpha$ 's are zero. Let  $M = [M_{hk}]$ , where  $M_{hk} (h, k = 1, 2, 3, 4)$  are the  $2 \times 2$  submatrices of  $M$ . Then by using the explicit forms of  $B_i$  and  $B_i B_j$ , we can write  $M$  as the sum of the following four matrices:

$$\begin{bmatrix} M_{11} & & & \\ & M_{22} & & \\ & & M_{33} & \\ & & & M_{44} \end{bmatrix} = \alpha_1 \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix} + \alpha_{23} \begin{bmatrix} -J & & & \\ & -J & & \\ & & J & \\ & & & -J \end{bmatrix} \\ + \alpha_{45} \begin{bmatrix} -J & & & \\ & J & & \\ & & -J & \\ & & & -J \end{bmatrix} + \alpha_{67} \begin{bmatrix} J & & & \\ & -J & & \\ & & -J & \\ & & & -J \end{bmatrix},$$

$$\begin{bmatrix} & M_{12} \\ M_{21} & \\ & M_{34} \\ & M_{43} \end{bmatrix} = \alpha_2 \begin{bmatrix} & K \\ -K & \\ & I \\ & -I \end{bmatrix} + \alpha_{13} \begin{bmatrix} & K \\ -K & \\ & \\ & -I \\ & I \end{bmatrix} \\
 + \alpha_3 \begin{bmatrix} & L \\ -L & \\ & J \\ & J \end{bmatrix} + \alpha_{12} \begin{bmatrix} & -L \\ L & \\ & \\ & J \\ & J \end{bmatrix} \\
 + \alpha_{46} \begin{bmatrix} & -I \\ I & \\ & -K \\ & K \end{bmatrix} + \alpha_{57} \begin{bmatrix} & -I \\ I & \\ & K \\ & -K \end{bmatrix} \\
 + \alpha_{47} \begin{bmatrix} & -J \\ -J & \\ & -L \\ & L \end{bmatrix} + \alpha_{56} \begin{bmatrix} & J \\ J & \\ & -L \\ & L \end{bmatrix},$$

$$\begin{bmatrix} & M_{13} \\ M_{31} & \\ & M_{24} \\ & M_{42} \end{bmatrix} = \alpha_4 \begin{bmatrix} & K \\ -K & \\ & -I \\ & I \end{bmatrix} + \alpha_{15} \begin{bmatrix} & K \\ -K & \\ & I \\ & -I \end{bmatrix} \\
 + \alpha_5 \begin{bmatrix} & L \\ -L & \\ & -J \\ & -J \end{bmatrix} + \alpha_{14} \begin{bmatrix} & -L \\ L & \\ & -J \\ & -J \end{bmatrix} \\
 + \alpha_{26} \begin{bmatrix} & I \\ -I & \\ & -K \\ & K \end{bmatrix} + \alpha_{37} \begin{bmatrix} & I \\ -I & \\ & K \\ & -K \end{bmatrix} \\
 + \alpha_{27} \begin{bmatrix} & J \\ J & \\ & -L \\ & L \end{bmatrix} + \alpha_{36} \begin{bmatrix} & -J \\ -J & \\ & -L \\ & L \end{bmatrix}$$



$$\begin{aligned}
\begin{bmatrix} & & & M_{14} \\ & & M_{23} & \\ & M_{32} & & \\ M_{41} & & & \end{bmatrix} &= \alpha_6 \begin{bmatrix} & & I \\ & K & \\ -I & -K & \end{bmatrix} + \alpha_{17} \begin{bmatrix} & & -I \\ & K & \\ I & -K & \end{bmatrix} \\
&+ \alpha_7 \begin{bmatrix} & J \\ & L \\ J & -L \end{bmatrix} + \alpha_{16} \begin{bmatrix} & J \\ & -L \\ J & L \end{bmatrix} \\
&+ \alpha_{24} \begin{bmatrix} & -K \\ & -I \\ K & I \end{bmatrix} + \alpha_{35} \begin{bmatrix} & K \\ & -I \\ -K & I \end{bmatrix} \\
&+ \alpha_{25} \begin{bmatrix} & -L \\ & -J \\ L & -J \end{bmatrix} + \alpha_{34} \begin{bmatrix} & -L \\ & J \\ L & J \end{bmatrix}.
\end{aligned}$$

Now,  $M = 0$  means that all its submatrices  $M_{hk}$  are zero. Since  $I, J, K, L$  are linearly independent, the equations  $M_{hk} = 0$  are equivalent to a number of linear equations in the  $\alpha$ 's, and from these linear equations we can easily see that the  $\alpha$ 's must all be zero. For example, it is obvious from the equations

$$M_{12} = (\alpha_2 + \alpha_{13})K + (\alpha_3 - \alpha_{12})L - (\alpha_{46} + \alpha_{57})I - (\alpha_{47} - \alpha_{56})J = 0,$$

$$M_{34} = (\alpha_2 - \alpha_{13})I + (\alpha_3 + \alpha_{12})J + (-\alpha_{46} + \alpha_{57})K - (\alpha_{47} + \alpha_{56})L = 0$$

that

$$\alpha_2, \quad \alpha_{13}, \quad \alpha_3, \quad \alpha_{12}, \quad \alpha_{46}, \quad \alpha_{57}, \quad \alpha_{47}, \quad \alpha_{56}$$

must all be zero. Thus we have proved that the 28 matrices  $B_i, B_i B_j$  are linearly independent.

(ii) Let  $G$  be the Lie subgroup of  $SO(8)$  generated by the elements  $B(\lambda')$ , and  $g$  its Lie algebra. Then  $g$  is a Lie subalgebra of the Lie algebra  $\mathfrak{o}(8)$  of  $SO(8)$ . We now prove that in fact  $g = \mathfrak{o}(8)$ .

From the theory of Lie groups we know that if  $t \rightarrow f(t)$ , where  $t \in \mathbb{R}$  and  $f(t) \in G$ , is any curve in  $G$  passing through the identity element

$I = f(0)$  of  $G$ , then the velocity vector  $f'(0)$  of this curve at  $I$  is an element of  $g$ . Now

$$t \rightarrow f_i(t) \equiv (\cos t)I + (\sin t)B_i \quad (i=1, \dots, 7)$$

are obviously curves in  $G$  such that  $f_i(0) = I$  and  $f'_i(0) = B_i$ . Therefore,  $B_i$  are all elements of  $g$ .

Since  $g$  is a Lie subalgebra of  $o(8)$  and  $B_i \in g$ , the Lie products  $[B_i, B_j] = B_i B_j - B_j B_i = 2B_i B_j$ , where  $i, j = 1, \dots, 7$ , and  $i < j$ , are all in  $g$ .

We have thus proved that the 28 linearly independent skew-symmetric matrices,  $B_i, B_i B_j$  all belong to  $g \subset o(8)$ . Since  $o(8)$  is the Lie algebra of all skew-symmetric matrices of order 8 and is therefore of dimension 28,  $g$  coincides with  $o(8)$ . This completes the proof of Theorem 2.6.

### 3. THE SPHERE BUNDLES $S^{2n-1} \rightarrow \Phi_n$ , $n = 2, 4$ , OR $8$ , WITH FIBERS ON MUTUALLY ISOCLINIC $n$ -PLANES IN $R^{2n}$

In  $R^{2n}$ ,  $n = 2, 4$ , or  $8$ , provided with rectangular coordinate system  $(x, y)$ , let  $S^{2n-1}$  be the unit sphere and  $\Phi_n$  the maximal set of mutually isoclinic  $n$ -planes  $\{x = 0, y = xB(\lambda)\}$  defined in Theorem 1.6. Then with the preparations we have made in § 2, we can now prove

**THEOREM 3.1.** *In  $R^{2n}$ ,  $n = 2, 4$ , or  $8$ , the  $n$ -planes in the maximal set  $\Phi_n$  of mutually isoclinic  $n$ -planes slice the unit sphere  $S^{2n-1}$  into a fiber bundle*

$$\mathcal{F}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n),$$

with base space  $\Phi_n$ , projection  $\pi$ , fiber  $S^{n-1}$  and group  $G_n$ , where  $G_2 = S^1$ ,  $G_4 = S^3$ , and  $G_8 = SO(8)$ .

*Proof.* We prove by exhibiting all the ingredients of a representative coordinate bundle.

(1) The bundle space  $S^{2n-1}$  has the equation  $xx^T + yy^T = 1$  in  $R^{2n}$ .

(2) The base space  $\Phi_n$  is covered by the two coordinate systems

$$(2.5) \quad (\Phi_n \setminus \mathbf{O}^\perp, \lambda), \quad (\Phi_n \setminus \mathbf{O}, \mu)$$

as in the proof of Theorem 2.3, where  $\mathbf{O}^\perp$  is the  $n$ -plane  $x = 0$ ,  $\mathbf{O}$  is the  $n$ -plane  $y = 0$ ,  $\lambda$  is the parameter in the equation  $y = xB(\lambda)$  of an  $n$ -plane in  $\Phi_n \setminus \mathbf{O}^\perp$ , and  $\mu$  is the parameter in the equation  $x = yB(\mu)^T$  of