Therefore, $f(\Psi_n)$ is the set Φ_n of mutually isoclinic n-planes in our Theorem 1.6.

2. Some further results

From now on we shall confine our attention to n-dimensional maximal sets of mutually isoclinic n-planes in R^{2n}, and therefore, n has always the values 2, 4, or 8 unless stated otherwise.

In this section, we prove a few more theorems for use in § 3. In these theorems, the indices a, b have the range of values $(0, 1, ..., n-1)$; $B_0 = I$ is the identity matrix of order n; $B_1, ..., B_{n-1}$ are the $n \times n$ matrices listed in Theorems 1.5 and 1.6; $\lambda = (\lambda_a)$ is an ordered set of n real parameters; and

$$B(\lambda) \equiv \sum_a \lambda_a B_a, \qquad N(\lambda) \equiv \sum_a \lambda_a^2 .$$

Moreover, for any matrix M, we denote its transpose by M^T.

THEOREM 2.1.

(i) $B(\lambda)B(\lambda)^T = N(\lambda)I$.

(ii) *If* $\lambda \neq 0$, *then*

$$B(\lambda)^{-1} = B(\lambda)^T/N(\lambda) = \sum_a \lambda_a B_a^T/N(\lambda),$$

so that if $\lambda \neq 0$, *the equation* $y = xB(\lambda)$ *is equivalent to the equation* $x = yB(\mu)^T$, *where* $\mu = \lambda/N(\lambda) \neq 0$.

(iii)
$$\det B(\lambda) = +\left(N(\lambda)\right)^{n/2} .$$

(iv) *If* $N(\lambda) = 1$, *then* $B(\lambda) \in SO(n)$, *where* $SO(n)$ *is the set of all orthogonal matrices of order* n *and determinant* $+1$.

Proof.
$$\begin{aligned} B(\lambda)B(\lambda)^T &= \left(\sum_a \lambda_a B_a\right)\left(\sum_b \lambda_b B_b^T\right) = \sum_{a,b} \lambda_a \lambda_b B_a B_b^T \\ &= \sum_a \lambda_a^2 B_a B_a^T + \sum_{a<b} \lambda_a \lambda_b (B_a B_b^T + B_b B_a^T), \end{aligned}$$

which, on account of the Hurwitz matrix equations (1.2), is equal to $(\sum_a \lambda_a^2)I = N(\lambda)I$. This proves (i), and also (ii). To prove (iii), we first note that since $B(\lambda)$ is a square matrix of order n, $\det B(\lambda)$ is a homogeneous polynomial of degree n in the λ_a's, and it follows from (i) that

$$(\det B(\lambda))^2 = \det\left(B(\lambda)B(\lambda)^T\right) = \left(N(\lambda)\right)^n .$$

Therefore,

$$\text{(2.1)} \qquad \det B(\lambda) = \pm\,(N(\lambda))^{n/2} = \pm\,(\lambda_0^2+\lambda_1^2+\dots+\lambda_{n-1}^2)^{n/2}$$
$$= \pm\,(\lambda_0^n + \text{other product terms in } \lambda_a)\,.$$

On the other hand, since $B_0 = I$, and $B_1, \dots, B_{n-1}$ are all skew-symmetric matrices, the diagonal elements of $B(\lambda)$ are all equal to λ_0, and none of the other elements of $B(\lambda)$ is equal to λ_0. Therefore,

$$\det B(\lambda) = \lambda_0^n + \text{other product terms in } \lambda_a\,.$$

Comparison of this with (2.1) gives (iii). Finally, (iv) follows immediately from (i) and (iii).

Returning to Theorems 1.2 and 1.6, we now prove

THEOREM 2.2. *Let* Φ_n *be the maximal set of mutually isoclinic n-planes in* R^{2n} *described in Theorem 1.6, and let* (u, v) *be any vector in* R^{2n}. *If* $u \neq 0$, *then the unique n-plane in* Φ_n *containing* (u, v) *is*

$$\text{(2.2)} \qquad y = x[vu^T-(vB_1u^T)B_1-\dots-(vB_{n-1}u^T)B_{n-1}]/(uu)^T\,.$$

If $v \neq 0$, *then the unique n-plane in* Φ_n *containing* (u, v) *is*

$$\text{(2.3)} \qquad x = y[uv^T-(uB_1^Tv^T)B_1^T-\dots-(uB_{n-1}^Tv^T)B_{n-1}^T]/(vv)^T\,.$$

Here, $B_1, \dots, B_{n-1}$ *are the matrices in (1.3), (1.4), or (1.5) according as* $n = 2, 4,$ *or* 8.

Proof. We shall prove only (2.2) for the case $u \neq 0$, as (2.3) for the case $v \neq 0$ can be proved similarly. Suppose that $u \neq 0$ and

$$\text{(2.4)} \qquad y = x(\lambda_0+\lambda_1B_1+\dots+\lambda_{n-1}B_{n-1})$$

is an n-plane in Φ_n containing (u, v). Then we have

$$v = u(\lambda_0+\lambda_1B_1+\dots+\lambda_{n-1}B_{n-1})\,,$$

which can be written as

$$v = [\lambda_0\lambda_1 \dots \lambda_{n-1}] \begin{bmatrix} u \\ \\ uB_1 \\ \vdots \\ uB_{n-1} \end{bmatrix}\,.$$

Multiplying the two sides of this equation on the right by

$$[u^T, -B_1u^T, ..., -B_{n-1}u^T]$$

and making use of the Hurwitz matrix equations (1.2), we get

$$v[u^T, -B_1u^T, ..., -B_{n-1}u^T] = [\lambda_0\lambda_1 \dots \lambda_{n-1}]\,(uu^T)I\,.$$

Since $uu^T \neq 0$, the above equation determines the λ_a's uniquely in terms of u, v. Now with these values of λ_a's, equation (2.4) becomes equation (2.2), as we wanted to prove. Incidentally, the above proof also confirms that there is exactly one n-plane in Φ_n containing the vector (u, v) (cf. Theorem 1.2).

Next, we give a direct proof of Theorem 1.3 for the special cases $n = 2, 4, 8$, and state the result as

THEOREM 2.3. *The maximal set* $\Phi_n = \{x = 0, y = xB(\lambda)\}$ *of mutually isoclinic n-planes in* R^{2n}, $n = 2, 4,$ *or* 8, *can be given a differentiable structure so that it is diffeomorphic with the n-sphere* S^n.

Proof. Let us regard Φ_n as a point set whose elements are the n-planes in Φ_n. Then, the subset $\Phi_n\backslash\mathbf{O}^\perp = \{y = xB(\lambda)\}$ of Φ_n is an open subset in which we can define a coordinate system by assigning to the element $y = xB(\lambda)$ the coordinate $\lambda = (\lambda_0, \lambda_1, ..., \lambda_{n-1})$. The subset $\Phi_n\backslash\mathbf{O} = \{x = 0$ and $y = xB(\lambda)$, where $\lambda \neq 0\}$ of Φ_n is also an open subset. By Theorem 2.1 (ii), this subset is the same as the subset $\{x = yB(\mu)^T\}$, and so, we can define in it a coordinate system by assigning to the element $x = yB(\mu)^T$ the coordinate $\mu = (\mu_0, \mu_1, ..., \mu_{n-1})$. Thus Φ_n is covered by the two coordinate neighborhoods

$$(\Phi_n\backslash\mathbf{O}^\perp, \lambda)\,, \quad (\Phi_n\backslash\mathbf{O}, \mu)\,. \tag{2.5}$$

Moreover, we can see from Theorem 2.1 (ii) that for any element in $(\Phi_n\backslash\mathbf{O}^\perp) \cap (\Phi_n\backslash\mathbf{O}) = \Phi_n\backslash\{\mathbf{O}^\perp, \mathbf{O}\}$, its two coordinates λ, μ, both nonzero, are related by

$$\mu = \lambda/N(\lambda), \text{ or equivalently, } \lambda = \mu/N(\mu)\,. \tag{2.6}$$

Hence, Φ_n is an n-dimensional manifold.

To show that Φ_n is diffeomorphic with the n-sphere S^n, we view S^n as the unit sphere $x_1^2 + ... + x_{n+1}^2 = 1$ in R^{n+1}, and use stereographic projections. Let $q_1(0, ..., 0, 1)$ and $q_2(0, ..., 0, -1)$ be respectively the north and south poles of S^n. Then S^n is the union of the two open subsets

$S^n \backslash q_1$ and $S^n \backslash q_2$. For an arbitrary point q in $S^n \backslash q_1$, let the line $q_1 q$ meet the equator n-plane $x_{n+1} = 0$ at the point $(\lambda, 0)$; and for an arbitrary point q in $S^n \backslash q_2$, let the line $q_2 q$ meet the equator n-plane $x_{n+1} = 0$ at the point $(\mu, 0)$. Then S^n is covered by the two coordinate neighborhoods

$$(S^n \backslash q_1, \lambda), \quad (S^n \backslash q_2, \mu). \tag{2.7}$$

Moreover, it is easy to verify that for a point in $S^n \backslash \{q_1, q_2\}$, its two coordinates λ and μ are also both nonzero and related by (2.6).

It now follows from (2.5), (2.6) and (2.7) that if f_1 is the map from $\Phi_n \backslash \mathbf{O}^{\perp}$ to $S^n \backslash q_1$ sending an n-plane in $\Phi_n \backslash \mathbf{O}^{\perp}$ with coordinate λ to the point in $S^n \backslash q_1$ with the same coordinate λ, and f_2 is the map from $\Phi_n \backslash \mathbf{O}$ to $S^n \backslash q_2$ sending an n-plane in $\Phi_n \backslash \mathbf{O}$ with coordinate μ to the point in $S^n \backslash q_2$ with the same coordinate μ, then f_1, f_2 combined will give a diffeomorphism from Φ_n to S^n.

In the remainder of this section, we are concerned exclusively with the matrices $B(\lambda)$ with $N(\lambda) = 1$. For convenience, we shall denote such matrices by $B(\lambda')$, with the understanding that λ' always satisfies the condition $N(\lambda') = 1$.

We know from Theorem 2.1 (iv) that every $B(\lambda')$ belongs to $SO(n)$. Let us now regard $SO(n)$ as the special orthogonal group. Then the set of elements $B(\lambda')$ of $SO(n)$ will generate a subgroup of $SO(n)$. We wish to know what this subgroup of $SO(n)$ is, and the next three theorems will give us the answer.

THEOREM 2.4. *For $n = 2$, the set of elements $B(\lambda')$ forms the group $SO(2)$ which is isomorphic with S^1.*

Proof. Since

$$B(\lambda') = \begin{bmatrix} \lambda_0' & \lambda_1' \\ -\lambda_1' & \lambda_0' \end{bmatrix} \quad \text{and} \quad \det B(\lambda') = (\lambda_0')^2 + (\lambda_1')^2 = 1,$$

the elements of $SO(2)$ are the elements $B(\lambda')$ themselves.

THEOREM 2.5. *For $n = 4$, the set of elements $B(\lambda')$ forms a 3-parameter subgroup of $SO(4)$, isomorphic with S^3.*

Proof. First, since $N(\lambda') = (\lambda'_0)^2 + \ldots + (\lambda'_3)^2 = 1$, the set $B(\lambda')$, with a natural topology, is homeomorphic with the unit 3-sphere S^3 in R^4. Next, using (1.4), we can easily verify that

$$B_2B_3 = -B_1, \quad B_3B_1 = -B_2, \quad B_1B_2 = -B_3.$$

With this and Theorem 2.1 (ii), straight forward computation will show that for any two elements $B(\lambda')$ and $B(\mu')$ of $SO(4)$, the product $B(\lambda')B(\mu')^{-1}$ is an element of $SO(4)$ of the form $B(\nu')$, where the components of ν' are analytic functions of the components of λ' and μ'. This proves our theorem.

For the case $n = 8$, we first observe that the elements $B(\lambda')$ of $SO(8)$ do not, by themselves, form a subgroup of $SO(8)$. For example, although B_1, B_2 are both of the form $B(\lambda')$, their product B_1B_2 is not. In fact, we have

THEOREM 2.6. *For $n = 8$, the set of elements $B(\lambda')$ of $SO(8)$ generates the group $SO(8)$ itself.*

Proof. Our proof consists of two steps (i) and (ii). In (i), we prove that the 28 skew-symmetric 8×8 matrices B_i, $B_iB_j (i, j = 1, \ldots, 7$, and $i<j)$ are linearly independent. In (ii), we prove that the Lie algebra of the subgroup of $SO(8)$ generated by the elements $B(\lambda')$ coincides with the Lie algebra $o(8)$ of $SO(8)$. The assertion in our theorem then follows from the well-known fact in Lie groups that there is a one-one correspondence between the connected Lie subgroups of a Lie group G and the Lie subalgebras of the Lie algebra of G.

(i) From (1.5), we see that the 8×8 matrices $B_i (i = 1, \ldots, 7)$ can be partitioned as

$$B_1 = \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix}, \quad B_2 = \begin{bmatrix} & K & & \\ -K & & & \\ & & & I \\ & & -I & \end{bmatrix}, \quad B_3 = \begin{bmatrix} & L & & \\ -L & & & \\ & & & J \\ & & J & \end{bmatrix}$$

$$B_4 = \begin{bmatrix} & & K & \\ & & & -I \\ -K & & & \\ & I & & \end{bmatrix}, \quad B_5 = \begin{bmatrix} & & L & \\ & & & -J \\ -L & & & \\ & -J & & \end{bmatrix},$$

$$B_6 = \begin{bmatrix} & & & I \\ & & K & \\ & -K & & \\ -I & & & \end{bmatrix}, \quad B_7 = \begin{bmatrix} & & & J \\ & & L & \\ & -L & & \\ J & & & \end{bmatrix},$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

are 2×2 submatrices and each empty space represents a 2×2 zero-matrix 0.

Since the matrices I, J, K, L have the properties

$$I^2 = I, \quad J^2 = -I, \quad K^2 = I, \quad L^2 = I,$$
$$JK = -KJ = -L, \quad KL = -LK = J, \quad LJ = -JL = -K,$$

we can easily verify that the products $B_i B_j (i, j = 1, \ldots, 7, \text{ and } i<j)$ are matrices of the same form as B_i, having some of O, $\pm I$, $\pm J$, $\pm K$, $\pm L$ as 2×2 submatrices.

To prove that the 28 matrices B_i, $B_i B_j$ are linearly independent, we construct the 8×8 matrix

$$M \equiv \textstyle\sum_i \alpha_i B_i + \sum\limits_{i<j} \alpha_{ij}(B_i B_j),$$

where the α's are some real numbers, and show that if $M = 0$, then all the α's are zero. Let $M = [M_{hk}]$, where $M_{hk}(h, k = 1, 2, 3, 4)$ are the 2×2 submatrices of M. Then by using the explicit forms of B_i and $B_i B_j$, we can write M as the sum of the following four matrices:

$$\begin{bmatrix} M_{11} & & & \\ & M_{22} & & \\ & & M_{33} & \\ & & & M_{44} \end{bmatrix} = \alpha_1 \begin{bmatrix} J & & & \\ & J & & \\ & & J & \\ & & & -J \end{bmatrix} + \alpha_{23} \begin{bmatrix} -J & & & \\ & -J & & \\ & & J & \\ & & & -J \end{bmatrix}$$

$$+ \alpha_{45} \begin{bmatrix} -J & & & \\ & J & & \\ & & -J & \\ & & & -J \end{bmatrix} + \alpha_{67} \begin{bmatrix} J & & & \\ & -J & & \\ & & -J & \\ & & & -J \end{bmatrix},$$

$$
\begin{bmatrix} & M_{12} & & \\ M_{21} & & & \\ & & & M_{34} \\ & & M_{43} & \end{bmatrix}
= \alpha_2 \begin{bmatrix} & K & & \\ -K & & & \\ & & & I \\ & & -I & \end{bmatrix}
+ \alpha_{13} \begin{bmatrix} & K & & \\ -K & & & \\ & & & -I \\ & & I & \end{bmatrix}
$$

$$
+ \alpha_3 \begin{bmatrix} & L & & \\ -L & & & \\ & & & J \\ & & J & \end{bmatrix}
+ \alpha_{12} \begin{bmatrix} & -L & & \\ L & & & \\ & & & J \\ & & J & \end{bmatrix}
$$

$$
+ \alpha_{46} \begin{bmatrix} & -I & & \\ I & & & \\ & & & -K \\ & & K & \end{bmatrix}
+ \alpha_{57} \begin{bmatrix} & -I & & \\ I & & & \\ & & & K \\ & & -K & \end{bmatrix}
$$

$$
+ \alpha_{47} \begin{bmatrix} & -J & & \\ -J & & & \\ & & & -L \\ & & L & \end{bmatrix}
+ \alpha_{56} \begin{bmatrix} & J & & \\ J & & & \\ & & & -L \\ & & L & \end{bmatrix},
$$

$$
\begin{bmatrix} & & M_{13} & \\ & & & M_{24} \\ M_{31} & & & \\ & M_{42} & & \end{bmatrix}
= \alpha_4 \begin{bmatrix} & & K & \\ & & & -I \\ -K & & & \\ & I & & \end{bmatrix}
+ \alpha_{15} \begin{bmatrix} & & K & \\ & & & I \\ -K & & & \\ & -I & & \end{bmatrix}
$$

$$
+ \alpha_5 \begin{bmatrix} & & L & \\ & & & -J \\ -L & & & \\ & -J & & \end{bmatrix}
+ \alpha_{14} \begin{bmatrix} & & -L & \\ & & & -J \\ L & & & \\ & -J & & \end{bmatrix}
$$

$$
+ \alpha_{26} \begin{bmatrix} & & I & \\ & & & -K \\ -I & & & \\ & K & & \end{bmatrix}
+ \alpha_{37} \begin{bmatrix} & & I & \\ & & & K \\ -I & & & \\ & -K & & \end{bmatrix}
$$

$$
+ \alpha_{27} \begin{bmatrix} & & J & \\ & & & -L \\ J & & & \\ & L & & \end{bmatrix}
+ \alpha_{36} \begin{bmatrix} & & -J & \\ & & & -L \\ -J & & & \\ & L & & \end{bmatrix}
$$

$$
\begin{aligned}
\begin{bmatrix} & & & M_{14} \\ & & M_{23} & \\ & M_{32} & & \\ M_{41} & & & \end{bmatrix}
&= \alpha_6 \begin{bmatrix} & & & I \\ & & K & \\ & -K & & \\ -I & & & \end{bmatrix}
+ \alpha_{17} \begin{bmatrix} & & & -I \\ & & K & \\ & -K & & \\ I & & & \end{bmatrix} \\
&+ \alpha_7 \begin{bmatrix} & & & J \\ & & L & \\ & -L & & \\ J & & & \end{bmatrix}
+ \alpha_{16} \begin{bmatrix} & & & J \\ & & -L & \\ & L & & \\ J & & & \end{bmatrix} \\
&+ \alpha_{24} \begin{bmatrix} & & & -K \\ & & -I & \\ & I & & \\ K & & & \end{bmatrix}
+ \alpha_{35} \begin{bmatrix} & & & K \\ & & -I & \\ & I & & \\ -K & & & \end{bmatrix} \\
&+ \alpha_{25} \begin{bmatrix} & & & -L \\ & & -J & \\ & -J & & \\ L & & & \end{bmatrix}
+ \alpha_{34} \begin{bmatrix} & & & -L \\ & & J & \\ & J & & \\ L & & & \end{bmatrix} .
\end{aligned}
$$

Now, $M = 0$ means that all its submatrices M_{hk} are zero. Since I, J, K, L are linearly independent, the equations $M_{hk} = 0$ are equivalent to a number of linear equations in the α's, and from these linear equations we can easily see that the α's must all be zero. For example, it is obvious from the equations

$$
\begin{aligned}
M_{12} &= (\alpha_2+\alpha_{13})K + (\alpha_3-\alpha_{12})L - (\alpha_{46}+\alpha_{57})I - (\alpha_{47}-\alpha_{56})J = 0, \\
M_{34} &= (\alpha_2-\alpha_{13})I + (\alpha_3+\alpha_{12})J + (-\alpha_{46}+\alpha_{57})K - (\alpha_{47}+\alpha_{56})L = 0
\end{aligned}
$$

that

$$\alpha_2, \quad \alpha_{13}, \quad \alpha_3, \quad \alpha_{12}, \quad \alpha_{46}, \quad \alpha_{57}, \quad \alpha_{47}, \quad \alpha_{56}$$

must all be zero. Thus we have proved that the 28 matrices B_i, B_iB_j are linearly independent.

(ii) Let G be the Lie subgroup of $SO(8)$ generated by the elements $B(\lambda')$, and g its Lie algebra. Then g is a Lie subalgebra of the Lie algebra $o(8)$ of $SO(8)$. We now prove that in fact $g = o(8)$.

From the theory of Lie groups we know that if $t \to f(t)$, where $t \in R$ and $f(t) \in G$, is any curve in G passing through the identity element

$I = f(0)$ of G, then the velocity vector $f'(0)$ of this curve at I is an element of g. Now

$$t \to f_i(t) \equiv (\cos t)I + (\sin t)B_i \qquad (i=1, ..., 7)$$

are obviously curves in G such that $f_i(0) = I$ and $f_i'(0) = B_i$. Therefore, B_i are all elements of g.

Since g is a Lie subalgebra of $o(8)$ and $B_i \in g$, the Lie products $[B_i, B_j] = B_iB_j - B_jB_i = 2B_iB_j$, where $i, j = 1, ..., 7$, and $i < j$, are all in g.

We have thus proved that the 28 linearly independent skew-symmetric matrices, B_i, B_iB_j all belong to $g \subset o(8)$. Since $o(8)$ is the Lie algebra of all skew-symmetric matrices of order 8 and is therefore of dimension 28, g coincides with $o(8)$. This completes the proof of Theorem 2.6.

3. The sphere bundles $S^{2n-1} \to \Phi_n$, $n = 2, 4$, or 8, with fibers on mutually isoclinic n-planes in R^{2n}

In R^{2n}, $n = 2, 4$, or 8, provided with rectangular coordinate system (x, y), let S^{2n-1} be the unit sphere and Φ_n the maximal set of mutually isoclinic n-planes $\{x = 0,\ y = xB(\lambda)\}$ defined in Theorem 1.6. Then with the preparations we have made in § 2, we can now prove

Theorem 3.1. *In R^{2n}, $n = 2, 4$, or 8, the n-planes in the maximal set Φ_n of mutually isoclinic n-planes slice the unit sphere S^{2n-1} into a fiber bundle*

$$\mathscr{I}_n = (S^{2n-1}, \Phi_n, \pi, S^{n-1}, G_n),$$

with base space Φ_n, projection π, fiber S^{n-1} and group G_n, where $G_2 = S^1$, $G_4 = S^3$, and $G_8 = SO(8)$.

Proof. We prove by exhibiting all the ingredients of a representative coordinate bundle.

(1) The bundle space S^{2n-1} has the equation $xx^T + yy^T = 1$ in R^{2n}.

(2) The base space Φ_n is covered by the two coordinate systems

$$(\Phi_n \backslash \mathbf{O}^\perp, \lambda), \quad (\Phi_n \backslash \mathbf{O}, \mu) \tag{2.5}$$

as in the proof of Theorem 2.3, where $\mathbf{O}^\perp$ is the n-plane $x = 0$, $\mathbf{O}$ is the n-plane $y = 0$, λ is the parameter in the equation $y = xB(\lambda)$ of an n-plane in $\Phi_n \backslash \mathbf{O}^\perp$, and μ is the parameter in the equation $x = yB(\mu)^T$ of