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ISOCLINIC n -PLANES IN R^{2n} AND THE HOPF-STEENROD
SPHERE BUNDLES $S^{2n-1} \rightarrow S^n$, $n = 2, 4, 8$

by Yung-Chow WONG and Kam-Ping MOK

0. INTRODUCTION

The construction of the sphere bundles $S^{2n-1} \rightarrow S^n$, $n = 2, 4, 8$, by N. Steenrod was accomplished in an ingenious but rather roundabout way, using the famous Hopf maps and the systems of complex numbers, quaternions and Cayley numbers (cf. Hopf [2], Steenrod [5, pp. 105-110] and Hilton [1, pp. 51-55]). In this paper, we show how the theory of mutually isoclinic n -planes in a real Euclidean $2n$ -space R^{2n} as developed by Wong in [8, 9] enables us to reconstruct these sphere bundles in a more natural manner by working strictly within the field of real numbers and giving the three cases $n = 2, 4, 8$ a more unified treatment. In addition, we prove that the bundle group $O(8)$ of the Hopf-Steenrod sphere bundle $S^{15} \rightarrow S^8$ can be replaced by $SO(8)$ but not by any subgroup of $SO(8)$.

In § 1, we recall certain results on maximal sets of mutually isoclinic n -planes in R^{2n} that motivated our investigation. In § 2, we confine ourselves to the cases $n = 2, 4, 8$, and prove some results that will be used later. In § 3, we construct three sphere bundles by using maximal sets of mutually isoclinic n -planes in R^{2n} . In § 4, we give a unified and explicit formulation of the three Hopf-Steenrod sphere bundles, using as Steenrod did the Hopf maps and systems of complex numbers, quaternions and Cayley numbers. In § 5, we prove that the Hopf maps and maximal sets of mutually isoclinic n -planes in R^{2n} , $n = 2, 4, 8$, are equivalent concepts, and that the reformulated Hopf-Steenrod sphere bundles described in § 4 are topologically essentially the same as the sphere bundles constructed in § 3. The paper ends with two appendices in which we explain the operations of Cayley numbers, and give a direct proof that for $n = 2, 4$, or 8 , the n -planes in R^{2n} containing the Hopf fibers of S^{2n-1} are mutually isoclinic n -planes.

In a continuation of this paper being prepared, we shall show that the image of the Hopf fibers of S^{2n-1} , $n = 2, 4$, or 8 , under an inversion in R^{2n} has some very interesting properties which include those recently found by J. B. Wilker [7] for the case $n = 2$.

We wish to thank Prof. Wilker for letting us have a preprint of his paper, and Prof. Kee-Yuen Lam for some helpful discussions.

1. SOME RESULTS ON ISOCLINIC n -PLANES IN R^{2n}

By a Euclidean (vector) m -space R^m , where m is a positive integer, we mean an m -dimensional vector space provided with a positive definite inner product. An r -plane ($1 \leq r \leq m-1$) in R^m is an r -dimensional vector subspace of R^m provided with the induced inner product. In R^m , length of a vector, angle between two vectors, orthogonality between a k -plane and an r -plane, (orthogonal) projection of a vector on an r -plane, orthonormal bases and rectangular coordinates are defined in the usual way.

In an R^{2n} , let \mathbf{A} , \mathbf{B} be any two n -planes. Then we say that \mathbf{A} is *isoclinic* with \mathbf{B} at angle θ if the angle between every nonzero vector in \mathbf{A} and its projection on \mathbf{B} is always equal to θ . It turns out that if \mathbf{A} is isoclinic with \mathbf{B} at angle θ , then \mathbf{B} is isoclinic with \mathbf{A} at the same angle θ . Therefore, in this case, we shall say that \mathbf{A} and \mathbf{B} are isoclinic at angle θ , or simply, \mathbf{A} and \mathbf{B} are isoclinic.

A set Φ of n -planes in R^{2n} is said to be a *maximal set of mutually isoclinic n -planes* if every pair of n -planes in Φ are isoclinic and Φ is not contained in a larger set of mutually isoclinic n -planes. It is easy to see from definition that if \mathbf{A} is isoclinic with \mathbf{B} at angle θ , then its orthogonal complement \mathbf{A}^\perp is isoclinic with \mathbf{B} at angle $\frac{\pi}{2} - \theta$. Consequently, if Φ is any maximal set of mutually isoclinic n -planes in R^{2n} and $\mathbf{A} \in \Phi$, then $\mathbf{A}^\perp \in \Phi$.

In his memoir [8] Wong determined, for each n , the dimensions of the maximal sets of mutually isoclinic n -planes in R^{2n} , the number of non-congruent maximal sets of a given dimension, and explicit equations of the n -planes in any maximal set of mutually isoclinic n -planes containing a given n -plane.

In the following, we summarize some of his results related to the problem studied in this paper.

THEOREM 1.1. (Wong [8, pp. 25-26]). *In R^{2n} provided with a rectangular coordinate system $(x, y) \equiv ([x_1 \dots x_n], [x_{n+1} \dots x_{2n}])$, any maximal set Φ of mutually isoclinic n -planes containing the n -plane $\mathbf{O}: y = 0$ (and consequently, also the n -plane $\mathbf{O}^\perp: x = 0$) is congruent to the set of n -planes with equations*