Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 34 (1988)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: QUILLEN'S THEOREM ON BUILDINGS AND THE LOOPS ON A SYMMETRIC SPACE

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Kapitel: §5. The Loops on a Symmetric Space

DOI: https://doi.org/10.5169/seals-56592

(4.6) THEOREM. *Evaluation at 1 induces an isomorphism* $L_{alg}G \cap P_I \cong G_I$. *In particular,* $L_{alg}G \cap P_I$ *is a compact Lie group.*

Proof. We have seen that e maps $L_{alg}G \cap P_I$ onto G_I. The kernel is $\Omega_{alg}G \cap P_I$. But $\Omega_{alg}G$ acts freely on $\mathcal{S}_G$, and $L_{alg}G \cap P_I$ fixes Δ_I, so $\Omega_{alg}G \cap P_I = \{1\}$.

Remark. As always, I is a proper subset of $\tilde{S}$ in (4.6). Of course (4.6) also depends on our assumption that G is simple. Its discrete analogue is the fact that W_I is finite if $\tilde{W}$ is irreducible. (It may be helpful to consider the "discrete" versions of all the results of this section. For example, the discrete version of "$\Omega_{alg}G$ acts freely on B_G" is "the coroot lattice $\mathrm{Hom}(S^1, T)$ acts freely on t (the foundation of $\mathcal{B}_G$)"; of course the latter assertion is trivial).

Note that we have shown that $\mathcal{S}_G/\Omega_{alg}G = G$, and in fact the orbit map $\mathcal{S}_G \to G$ is given by evaluation at $t = 1$. This can also be proved directly. It reduces to the following interesting theorem, also observed by Quillen.

(4.7) THEOREM. *Suppose* $X, Y \in \mathfrak{g}$ *and* $\exp X = \exp Y$. *Then* $\exp tX = f(e^{2\pi it}) \exp tY$ *for some* $f \in \Omega_{alg}G$. □

It is not hard to prove this directly—for example, it is enough to prove it for $G = U(n)$. Not surprisingly, however, it is also implicit in what we have already one. First, one can reduce to the case when G is simple and simply-connected. Using (1.3), one can easily reduce further to the case $X \in \mathring{\Delta}_I$, $Y = g \cdot X$ for some $g \in G$. Then $g \in C_G \exp X = G_I$, so $g = h(1)$ with $h \in L_{alg}G \cap P_I$. Let $h = \exp tX\, g \exp -tX$; then $h \in L_{alg}G$ and $f = hh(1)^{-1}$ is the desired loop.

§ 5. THE LOOPS ON A SYMMETRIC SPACE

We assume throughout this section that G is simple and simply—connected. If σ is an involution on G with fixed group K, as usual, then K is connected and G/K is simply—connected. The notations and conventions of § 1 and § 3 remain in force.

The loop space $\Omega(G/K)$ is homotopy equivalent to the space of paths in G that start at the identity and end in K. Now consider the involution τ on ΩG given by $\tau(f)(z) = \sigma(f(\bar{z}))$. The fixed group $(\Omega G)^\tau$ is clearly homeomorphic to our space of paths, since $f \in (\Omega G)^\tau$ implies $f(-1) \in K$.

Henceforth we will always consider $(\Omega G)^\tau$ in place of $\Omega(G/K)$. Note also the definition of τ extends to LG, $LG_{\mathbf{C}}$, and even $L_{alg}G_{\mathbf{C}}$: for if $f: \mathbf{C}^* \to G_{\mathbf{C}}$ is a regular map, so is $\sigma \circ f \circ (z \mapsto \bar{z})$, since σ is anti—complex on $G_{\mathbf{C}}$.

(5.1) THEOREM (Quillen). *The inclusion* $(\Omega_{alg}G)^\tau \to (\Omega G)^\tau$ *is a homotopy equivalence.*

We defer the proof to the end of this section.

Thus $\Omega(G/K)$ can be thought of as a real form of $\Omega_{alg}G_{\mathbf{C}}$. More precisely, $(L_{alg}G_{\mathbf{C}})^\tau$ is a real form of $L_{alg}G_{\mathbf{C}}$, and $\Omega(G/K)$ is a homogeneous space of this real form. For clearly P (regular maps $\mathbf{C} \to G_{\mathbf{C}}$) is invariant under τ, so from (3.3) we obtain a corresponding "Iwasawa" decomposition.

(5.2) THEOREM. *The multiplication map* $(\Omega_{alg}G)^\tau \times P^\tau \to (L_{alg}G_{\mathbf{C}})^\tau$ *is an homeomorphism.* □

On the other hand $\tilde{B}$ is of course not τ-invariant in general, since B is not σ-invariant. However the parabolic subgroup $\tilde{Q}$ corresponding to the black nodes on the extended Satake diagram is clearly τ-invariant; in fact $\tilde{Q} = Q \times U^\#$, where $U^\# = \{f \in P: f(0) = 1\}$ (note $U^\#$ is τ-invariant). Now consider $\tilde{N}_{\mathbf{C}} = L_{alg}N_{\mathbf{C}}$. Since σ preserves $N_{\mathbf{C}}$, τ preserves $\tilde{N}_{\mathbf{C}}$. Note $\operatorname{Hom}(S^1, T)$ is also τ invariant and in fact if $f \in \hom(S^1, T)$, $\tau f = \sigma(f(z)^{-1})$. It follows that $(\hom(S^1, T))^\tau = \hom(S^1, T_{\mathfrak{m}}) \cong R_{\mathfrak{m}}$. It is also easy to see that $\tilde{N}^\tau_{\mathbf{C}} \cap \tilde{Q}$ is normal in $(\tilde{N}_{\mathbf{C}})^\tau$; the quotient is $\tilde{W}_{\mathbf{R}}$. Here we recall that $\tilde{W}_{\mathbf{R}}$ is the affine Weyl group associated to the restricted root system Σ; it has a canonical set of Coxeter generators $\tilde{S}_{\mathbf{R}}$. Write $\tilde{G}_{\mathbf{R}}$, $\tilde{B}_{\mathbf{R}}$, $\tilde{N}_{\mathbf{R}}$, for $(\tilde{G}_{\mathbf{C}})^\tau$, $\tilde{Q}^\tau$, $\tilde{N}^\tau_{\mathbf{C}}$, respectively.

(5.3) THEOREM. $(\tilde{G}_{\mathbf{R}}, \tilde{B}_{\mathbf{R}}, \tilde{N}_{\mathbf{R}}, \tilde{S}_{\mathbf{R}})$ *is a topological Tits system satisfying the four axioms (2.11), (2.12), (2.20) and (2.21).*

Before giving the proof, we discuss some corollaries. If $I \subset \tilde{S}_{\mathbf{R}}$, we let $\tilde{Q}_I$ denote the parabolic subgroup $P_{I'}$ of $\tilde{G}_{\mathbf{C}}$; here I' consists of the black nodes of the extend ended Satake diagram together with the white nodes that correspond under restriction to elements of I (for example, $\tilde{Q} = \tilde{Q}_\phi$). Then $\tilde{Q}_I$ is τ-invariant and the parabolic subgroups (containing $\tilde{Q}^\tau$) are precisely the subgroups $\tilde{Q}^\tau_I$. Let $\mathcal{O}_I = \tilde{Q}^\tau_I$. The proof of (5.3) will show that for the minimal parabolics $\mathcal{O}_s$, $s \in \tilde{S}_{\mathbf{R}}$, $\mathcal{O}_s/\tilde{B}_{\mathbf{R}}$ is sphere of dimension $n(s) \equiv m(\alpha_s) + m(2\alpha_s)$ (here the multiplicity $m(2\alpha_s)$ is of course zero if $2\alpha_s$ is not a root). If $s_1 \ldots s_k$ is a reduced decomposition of $w \in W^I$, let $n(w) = n(s_1) + \cdots + n(s_k)$.

(5.4) COROLLARY. *The Bruhat decomposition of* $\tilde{G}_{\mathbf{R}}/\mathcal{O}_I$ *is a* CW *decomposition, and the closure relations on the cells are given by the Bruhat order on* $\tilde{W}^I_{\mathbf{R}}$. *Furthermore the cell series is* $\sum_{w\in\tilde{W}^I_{\mathbf{R}}} t^{n(w)}$. □

(5.5) COROLLARY (Bott-Samelson). $\Omega G/K$ *has the homotopy type of a* CW-*complex with cell series* $\sum_{w\in\tilde{W}^I_{\mathbf{R}}} t^{n(w)}$, *where* $I = S_{\mathbf{R}}$. □

The cell series obtained by Bott and Samelson ([7], Corollary 3.10) is described in terms of the diagram for $\mathfrak{t}_{\mathfrak{m}}$, but can be shown to agree with the one above (cf. [25] for the case of ΩG). Bott and Samelson also showed that the cells they constructed are all cycles mod 2. Here, reverting temporarily to the notation of § 2, their result appears in the following form.

(5.6) THEOREM. *Let* (G, B, N, S) *be a topological Tits system satisfying the four axioms, and let* P *be a parabolic subgroup. Then the Bruhat cells of* G/P *are all cycles mod 2.*

Proof. Let $P = P_I$, $I \leqslant S$, and fix $w \in W^I$. Let $s_1 \dots s_k$ be a reduced decomposition of w. If $k = 1$ then P_{s_1}/B is a sphere and maps homeomorphically onto $\bar{E}_{s_1}$ by $xB \mapsto xP$. Hence E_w is an integral cycle. In general, consider the space $X_w = P_{s_1} \times_B P_{s_2} \times_B \dots \times_B P_{s_k}/B$, and let $w' = s_2 \dots s_k$. By assumption each projection $P_s \to P_s/B$ is a locally trivial principal B-bundle, so the natural projection $X_w \to P_{s_1}/B$ is a locally trivial fibre bundle with fibre $X_{w'}$. Hence we conclude by induction on k that X_w is a topological manifold (not necessarily orientable). The fundamental class in mod 2 homology is represented by the cell $A_{s_1} \times A_{s_2} \dots \times A_{s_k}$ in X_w, where $A_s \leqslant P_s$ is chosen as in the proof of theorem 2.22, and by the Steinberg lemma (2.9) this cell is carried homeomorphically onto E_w under the natural (multiplication) map $X_w \to G/P$. This proves the theorem. □

Returning to our standard notation, we have:

(5.7) COROLLARY (Bott-Samelson). $\Omega G/K$ *has* mod 2 *Poincaré series as in (5.5).* □

In general one could ask for a combinatorial formula describing the differential in the cellular chain complex: $\partial[E_w] = \sum_{x\to w} a_x[E_x]$, where the sum is over the $x \in W^I$ that immediately precede w in the Bruhat order, and satisfy $n(x) + 1 = n(w)$. The problem is to determine the integers a_x.

Of course if the multiplicities $m(\alpha_s), m(\alpha_{2s})$ are all even, every cell is an integral cycle. Here we recall that the multiplicities are all even if and only if G/K is of "splitting rank" (not to be confused with the split form mentioned earlier): that is, rank K + rank G/K = rank G. For example, G itself, regarded as a symmetric space, is of splitting rank, as is $SU(2n)/Sp(n)$.

(5.8) COROLLARY. *If G/K is of splitting rank, the integral homology of $\Omega G/K$ is concentrated in even dimensions, and the Poincaré series is given by the series of (5.5).*

The "somewhat mysterious application..." of Bott-Samelson ([7], 4.1) is quite transparent from the present point of view.

(5.9) THEOREM (Bott-Samelson). *Suppose* rank G/K = rank G *(i.e., $G_{\mathbf{R}}$ is the split real form of $G_{\mathbf{C}}$). Then* $\dim H_q(\Omega G/K, \mathbf{Z}/2) = \dim H_{2q}(\Omega G; \mathbf{Z}/2)$. *Hence the* mod 2 *Poincaré series of $\Omega G/K$ is $\prod_{i=1}^{l}(1-t^{m_i})^{-1}$, where the m_i are the exponents of G.*

Proof. By assumption, $\mathfrak{t}_{\mathfrak{m}} = \mathfrak{t}$. It follows at once that τ preserves $\tilde{B}$ and is the identity on $\tilde{W}$; hence τ preserves the Bruhat cells in $\tilde{G}_{\mathbf{C}}/P$. Furthermore, each cell is identified with a complex vector space in such a way that τ corresponds to a linear conjugation. Since every cell is a cycle mod 2, this proves the theorem. (In more detail, σ preserves the root subalgebras X_α, and of course acts anti-linearly. The same is true for τ acting on the $X_{n,\alpha}$, and hence (by definition) for τ acting on the root subgroups $\exp X_{n,\alpha}$. In particular τ acts by a conjugation on each U_s, $s \in \tilde{S}$. But every cell can be identified with a product of subgroups U_s, by the Steinberg lemma.) □

Remark. Bott and Samelson obtain similar results with $\Omega(G/K)$ replaced by suitable homogeneous spaces of K. For example, if rank G/K = rank G, they show that $\dim H_q(K/C_k t_m; \mathbf{Z}/2) = \dim H_{2q}(G/T, \mathbf{Z}/2)$. These results also fit neatly into the present context, using the topological Tits system $(G_{\mathbf{R}}, B_{\mathbf{R}}, N_{\mathbf{R}}, S_{\mathbf{R}})$. The points is that $G/T = G_{\mathbf{C}}/B$, $K/C_k t_m = G_{\mathbf{R}}/B_{\mathbf{R}}$, etc.

Proof of Theorem 5.3. Axiom (2.1) is easy and is left to the reader. The proof of the remaining three axioms for an ordinary Tits system follows a standard pattern and will only be sketched. The first step is to prove the Bruhat decomposition directly. One way of doing this, which is of some independent interest, is sketched in § 8. Briefly, the argument is as follows. The $\tilde{Q}$-orbits in $\tilde{G}_{\mathbf{C}}/\tilde{Q}$ are vector bundles over certain flag varieties, and

τ acts on each orbit as a conjugate linear bundle automorphism. For the orbit $\tilde{Q}w\tilde{Q}/\tilde{Q}$, this action is free on the base unless $w \in \tilde{W}_{\mathbf{R}}$. Furthermore, if $w \in \tilde{W}_{\mathbf{R}}$ then $\tilde{Q}\tilde{w}\tilde{Q} = \tilde{B}w\tilde{Q}$ so the Bruhat cell $\tilde{B}w\tilde{Q}/\tilde{Q}$ is τ-invariant. The Bruhat decomposition for $\tilde{G}_{\mathbf{R}}$ then follows by taking τ fixed points of the $\tilde{Q} - \tilde{Q}$ double coset decomposition of $\tilde{G}_{\mathbf{C}}$. In particular this proves that $\tilde{B}_{\mathbf{R}}$, $\tilde{N}_{\mathbf{R}}$ generate $\tilde{G}_{\mathbf{R}}$. Axiom (2.3) is easy. For (2.4), we use induction on $l(w)$. The inductive step reduces to showing that $s\tilde{B}_{\mathbf{R}}\,s \subseteq \tilde{B}_{\mathbf{R}} \cup \tilde{B}_{\mathbf{R}}\,s\tilde{B}_{\mathbf{R}}$, which in turn can be deduced from the Bruhat decomposition for rank one groups (already proved). (Cf. [33], Prop. 1.2.3.17, for the details of one version of this argument.)

Axiom (2.11) is immediate since $\tilde{W}_{\mathbf{R}}$ is an irreducible affine Weyl group (see § 3). For the remaining axioms, we need to explicitly construct certain subgroups $\tilde{K}_i$ (analogues of the "little $SU(2)$" subgroups in the loop group case), where $\tilde{K}_i$ corresponds to the ith simple root β_i of the affine restricted root system $\tilde{\Sigma}$. When $i \neq 0$, $\tilde{K}_i$ is the group of constant loops $K_{\beta i}$ already constructed in § 1. $\tilde{K}_0$ is constructed in the same way. Let $I \subseteq \tilde{S}$ be the subset formed by taking the union of the black nodes and the special node $-\alpha_0$ of the extended Satake diagram, and then taking the path component of $-\alpha_0$ in this smaller diagram. Let $\tilde{G}_I = L_{alg}G \cap P_I$ (compare § 4). Then $\tilde{G}_I$ and its commutator subgroup $\tilde{G}'_I$ are τ-invariant subgroups and we define $\tilde{K}_0 = (\tilde{G}'_I)^{\tau}$. Note that $\tilde{K}_0$ is a compact subgroup of $\tilde{G}_{\mathbf{R}}$; in fact evaluation at 1 yields an embedding $\tilde{K}_0 \to K$. (Note however that $\tilde{K}_0$ does not consist of K-valued loops.) The complexification of $\tilde{G}'_I$ is the subgroup $\tilde{G}'_{\mathbf{C},I}$ generated by the root subgroups U_i, $i \in I$. Passing to τ-fixed points we obtain a semisimple real form $\tilde{G}_{\mathbf{R},0}$ with $\tilde{K}_0$ as maximal compact. The structure of these groups is easily read off from the Satake diagram.

Example. Let $G = SU(4)$, $K = Sp(2)$, as in § 1. Then $\tilde{S} \cong (0, 1, 2, 3)$ and $I = (0, 1, 3)$. The parabolic P_I consists of all matrices $\begin{pmatrix} A & Bz \\ Cz^{-1} & D \end{pmatrix}$ in $\tilde{G}_{\mathbf{C}}$ with A, B, C, D 2×2 matrices over $\mathbf{C}[z]$. $\tilde{G}_{\mathbf{C},I}$ consists of the elements of P_I with A, B, C, D constant; note evaluation at one is in this case an isomorphism onto the constant loops. In this example $\tilde{G}_I = \tilde{G}'_I \cong SU(4)$ and $\tilde{K}_0$ is the subgroup of matrices as above with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2)$. In particular $\tilde{K}_0$ is isomorphic to $Sp(2)$; note this in fact follows immediately from the Satake diagram.

Now let $\mathcal{O}_i$ be the minimal parabolic $\langle \tilde{B}_{\mathbf{R}}, s_i \rangle \leqslant \tilde{G}_{\mathbf{R}}$, as usual. In Axiom (2.12) we take $A_i = \tilde{K}_i$. Certainly $\tilde{K}_i$ is compact and contains 1, and since

$\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}$ is a subgroup of lower dimension, we have $\tilde{K}_i = \overline{\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}\, s_i \tilde{B}_{\mathbf{R}}}$. The Iwasawa decomposition of $\tilde{G}_{\mathbf{R},i}$ shows that $\tilde{K}_i\tilde{B}_{\mathbf{R}} = \tilde{G}_{\mathbf{R},i}\tilde{B}_{\mathbf{R}}$. Now $\mathcal{O}_i = \mathbf{B}_{\mathbf{R}} \quad \mathbf{B}_{\mathbf{R}}\, s_i \mathbf{B}_{\mathbf{R}}$, and $B_{\mathbf{R}}\, s_i \mathbf{B}_{\mathbf{R}} = U_{\mathbf{R},i} s_i B_{\mathbf{R}}$, where $U_{\mathbf{R},i}$ corresponds to the positive roots β_i and (if $2\beta_i$ is a root) $2\beta_i$. Since $U_{\mathbf{R},i} \leqslant \tilde{G}_{\mathbf{R},i}$, this completes the proof of (2.12). Note $\mathcal{O}_i/\tilde{B}_{\mathbf{R}} = \tilde{K}_i/\tilde{K}_i \cap \tilde{B}_{\mathbf{R}}$. Since $U_{\mathbf{R},i}$ is homeomorphic to a real vector space of dimension $n_i = m_{\beta i} + m_{2\beta i}$, and $\mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ is compact, we also conclude that $\mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ is a sphere of dimension n_i, and that $\mathcal{O}_i \to \mathcal{O}_i/\tilde{B}_{\mathbf{R}}$ has a local section. This completes the proof of Theorem 5.3. □

Now let $\mathcal{B}_{G/K}$ be the building associated to the topological Tits system of (5.3). To prove Theorem 5.1, it is enough to show (as in § 4):

(5.4) THEOREM (Quillen). *$(\Omega_{alg}G)^\tau$ acts freely on $\mathcal{B}_{G/K}$, with orbit space G/K.*

Proof. $B_{G/K}$ is a quotient space of $(\Omega_{alg}G)^\tau \times K/C_K t_m \times \Delta$, where Δ is the Cartan simplex in $\mathfrak{t}_{\mathfrak{m}}$ (here we are using (5.2); note that $(L_{alg}G)^\tau \cap P^\tau = G^\sigma = K$). Hence the orbit space of the $(\Omega_{alg}G)^\tau$-action is a quotient of $K/C_K t_m \times \Delta$. As in the proof of (4.2), we see that the equivalence relation here coincides with that of Theorem 1.9. Hence the orbit space is G/K, as desired. To see that the action is free, we introduce the space of special paths $\mathcal{S}_{G/K}$ path of the form $f(e^{2\pi i t}) \exp tX$ with $f \in (\Omega_{alg}G)^\tau$ and $X \in \mathfrak{m}$. The proof now proceeds exactly as in (4.2); details are left to the reader. □

The other results of § 4 also go through: $\mathcal{S}_{G/K}$ is $(L_{alg}G)^\tau$—equivariantly homeomorphic to the building $\mathcal{B}_{G/K}$, and if $X, Y \in m$, $\exp X = \exp Y$ implies $\exp tX = f \exp tY$, where $f \in (\Omega_{alg}G)^\tau$.

§ 6. EXAMPLES

In this section we discuss six examples, the first four of which arise in the Bott periodicity theorems (§ 7). The first and last examples are discussed in some detail, the others are only sketched.

(6.1) $\Omega(SU(2n)/Sp(n))$. This is perhaps the simplest nonsplit example. $SU(2n)$ has an involution σ given by $\sigma(A) = J\bar{A}J^{-1}$, where J is the matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. The fixed group K is $Sp(n)$. The extension of σ to $SL(2n, \mathbf{C})$