and I is a proper subset of $\tilde{S}$, then $\tilde{W}_I$ is finite. This is obvious since the elements of I have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take $A_s = \tilde{G}_s$. We have $\tilde{G}_s\tilde{B} = \tilde{G}_{\mathbf{C},s}\tilde{B} = \tilde{B} \quad U_s s\tilde{B} = P_s$. In particular $P_s/\tilde{B} = \tilde{G}_s/(\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbf{C}P^1$, which also proves Axioms (2.20) and (2.21). □

(3.2) COROLLARY. *$\Omega_{alg}G$ is a CW-complex with cells of even dimension, indexed by $\operatorname{Hom}(S^1, T)$. The Poincaré series for its integral homology is $\sum_{\lambda \in \operatorname{Hom}(S^1,T)} t^{2\bar{l}(\lambda)}$, where $\bar{l}(\lambda)$ is the minimal length accuring in λW. Identifying $\operatorname{Hom}(S^1, T)$ with $\tilde{W}^S$, the closure relations on the cells are given by the Bruhat order on $\tilde{W}^S$.* □

Remark. An explicit formula for $\bar{l}(\lambda)$ is given in [16], Prop. 1.25: $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$.

We will also need the "Iwasawa decomposition" (see [17], [27], [29]):

(3.3) THEOREM. $\tilde{G}_{\mathbf{C}} = \Omega_{alg}G \times P$. □

Remark. Note that (3.3) shows that the associated building, which we will be denoted simply by $\mathscr{B}_G$, is a quotient of $L_{alg}G/T \times \Delta$. The equivalence relation is then $(f_1 T, X) \sim (f_2 T, X)$ if $X \in \mathring{\Delta}_I$ and $f_1 = f_2 \bmod LG \cap P_I$.

§ 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. *Let G be a compact Lie group. Then the inclusion $\Omega_{alg}G \to \Omega G$ is a homotopy equivalence.*

If G is simply connected, let $\mathscr{B}_G$ denote the topological building associated to the algebraic loop group $L_{alg}G_{\mathbf{C}}$ as in § 2.

(4.2) THEOREM (Quillen). *$\Omega_{alg}G$ acts freely on $\mathscr{B}_G$, with orbit space G.*

Proof of (4.1). It is easy to reduce to the case when G is simply connected. Since B_G is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that $\Omega_{alg}G \to \Omega G$ is a weak equivalence. Since both spaces have the homotopy type of a CW-complex, the map is in fact a homotopy equivalence. □

Since G is a product of simple groups (as is $G_{\mathbf{C}}$), it is very easy to reduce to the case when G is simple. For the rest of this section, then, we assume G is simple and simply-connected, of rank l.

To prove 4.2, we introduce Quillen's space of special paths $\mathscr{S}_G$: this is the space of all paths $[0, 1] \to G$ of the form $f(e^{2\pi i t}) \exp tX$, where $f \in \Omega_{alg}G$ and $X \in \mathfrak{g}$. $\mathscr{S}_G$ is topologized as a quotient of $\Omega_{alg}G \times \mathfrak{g}$. Note that $L_{alg}G$ acts on $\mathscr{S}_G$ by $h \cdot (f \exp tX) = hf \exp tXh(1)^{-1}$. The following key lemma, whose proof is deferred, also helps to explain the significance of the parabolic subgroups P_I.

(4.3) LEMMA. *Suppose* $X \in \mathring{\Delta}_I$, *then the isotropy group of* $\exp tX$ *is* $L_{alg}G \cap P_I$.

(4.4) THEOREM (Quillen). $\mathscr{S}_G$ *is* $L_{alg}G$*-equivariantly homeomorphic to the building* $\mathscr{B}_G$.

Proof. The action map $\varphi: L_{alg}G \times \Delta \to \mathscr{S}_G$ given by

$$\varphi(f, X) = f \exp tXf(1)^{-1}$$

is surjective by Theorem 1.1. If $\varphi(f_1, X_1) = \varphi(f_2, X_2)$, then (evaluating at $t=1$) $\exp X_1$ and $\exp X_2$ are conjugate in G, so $X_1 = X_2$ by Theorem 1.3. We then have $\varphi(f_1, X) = \varphi(f_2, X)$ if and only if $f_1 = f_2$ mod the isotropy group of $\exp tX$. Hence, by (4.3), φ factors through the desired homeomorphism $\mathscr{B}_G \to \mathscr{S}_G$. □

Remark. Here we have used the Iwasawa decomposition (3.3) to identify $\mathscr{B}_G = (\tilde{G}_{\mathbf{C}}/\tilde{B} \times \Delta)/\sim$ with $(L_{alg}G/T \times \Delta)/\sim$.

(4.5) LEMMA. $L_{alg}G \cap P_I$ *is generated by* T *and the subgroups* $\tilde{G}_i, i \in I$.

Proof. We have $P_I = \tilde{B}W_I\tilde{B}$. By the Steinberg lemma (2.9), each $\tilde{B}w\tilde{B} (w \in W_I)$ has the form XB, where X is a product of the $\tilde{G}_i$. Since $L_{alg}G \cap XB = XT$, the lemma follows. □

Proof of 4.2. The action of $\Omega_{alg}G$ on $\mathscr{S}_G$ is clearly free. By (4.4), the same is true for $\mathscr{B}_G$. Now consider the orbit space $\mathscr{B}_G/\Omega_{alg}G$. Since $\mathscr{B}_G = (L_{alg}G/T \times \Delta)/\sim\; = (\Omega_{alg}G \times G/T \times \Delta)/\sim$, the orbit space is a quotient of $G/T \times \Delta$. The equivalence relation is given by $(g_1T, X) \sim (g_2T, X)$ if $X \in \mathring{\Delta}_I$ and $g_2 = fg_1p$ with $f \in \Omega_{alg}G$, $p \in P_I$. In fact $p \in LG \cap P_I$. Now let $\bar{G}_I = e(LG \cap P_I)$, where e is evaluation at $z = 1$. Then $(g_1T, X) \sim (g_2T, X)$ if and only if $g_1 = g_2 \bmod \bar{G}_I$. For if $g_2 = f\, g_1\, p$ as above, then $\bar{G}_I = e(L_{alg}G \cap P_I)$, where e is evaluation at $z = 1$. Then $(g_1T, X) \sim (g_2T, X)$ if and only if $g_1 = g_2 \bmod \bar{G}_I$. For if $g_2 = f\, g_1\, p$ as above, then

$g_2 = f\, g_1\, p(1)$, and conversely if $g_2 = g_1\, p(1)$, then $g_2 = f\, g_1\, p$, where $f = g_2\, p^{-1}\, g_1^{-1}$. But by (4.5), $\bar{G}_I = G_I$ (see § 1). In other words, the equivalence relation here coincides with the classical relation of Theorem 1.5, which has quotient G. $\square$

Proof of 4.3. Fix $X \in \mathring{\Delta}_I$. We first show that $L_{alg}G \cap P_I$ fixes $\exp tX$ in $\mathcal{S}_G$. By (4.5) it is enough to show that each $\tilde{G}_i (i \in I)$ fixes

$$\exp tX : f(e^{2\pi i t}) \exp tX f(1)^{-1} = \exp tX .$$

If $i \neq 0$, $\tilde{G}_i = G_i$ is a subgroup of the constant loops, so f is a constant $g \in G_i$. The desired equation is then equivalent to $g \cdot X = X$ (recall that $g \cdot X = Ad(g)X$). But since $i \neq 0$, $\alpha_i(X) = 0$, so this is true by definition. Now suppose $i = 0$, so that X lies on the outer wall: $\alpha_0(X) = 1$. Then $X = \frac{1}{2}\alpha_0^* + Y$, where $\alpha_0^* = 2\alpha_0/\alpha_0 \cdot \alpha_0$ is the coroot of α_0 and $\alpha_0(Y) = 0$.

The equation we want can be written $(f \in \tilde{G}_0)$:

$$f(e^{2\pi i t}) = \exp tX\ f(1) \exp -tX$$

Since $f(1) \in G_0$, $f(1) \cdot Y = Y$, and our equation simplifies to

$$f(e^{2\pi i t}) = \exp\left(\frac{1}{2} t\alpha_0^*\right) f(1) \exp\left(-\frac{1}{2} t\alpha_0^*\right)$$

Note this is now an equation in the path space of G_0. Identifying G_0 with $SU(2)$, it can be written

$$\begin{pmatrix} a & be^{2\pi i t} \\ ce^{-2\pi i t} & d \end{pmatrix} = \begin{pmatrix} e^{\pi i t} & 0 \\ o & e^{-\pi i t} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{-\pi i t} & o \\ o & e^{\pi i t} \end{pmatrix}$$

Where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$. This last equation is obviously correct, and we conclude that $L_{alg}G \cap P_I$ fixes $\exp tX$.

Conversely, suppose

$$f \exp tX f(1)^{-1} = \exp tX , \quad \text{or} \quad f = \exp tX\ f(1) \exp(-tX) .$$

Then $f(1) \in C_G \exp X = G_I$, and hence $f(1) = h(1)$ for some $h \in L_{alg}G \cap P_I$. But then $h = \exp tXh(1) \exp -tX = f$.

A useful fact that follows from all this is:

(4.6) THEOREM. *Evaluation at 1 induces an isomorphism* $L_{alg}G \cap P_I \cong G_I$. *In particular,* $L_{alg}G \cap P_I$ *is a compact Lie group.*

Proof. We have seen that e maps $L_{alg}G \cap P_I$ onto G_I. The kernel is $\Omega_{alg}G \cap P_I$. But $\Omega_{alg}G$ acts freely on $\mathscr{S}_G$, and $L_{alg}G \cap P_I$ fixes Δ_I, so $\Omega_{alg}G \cap P_I = \{1\}$.

Remark. As always, I is a proper subset of $\tilde{S}$ in (4.6). Of course (4.6) also depends on our assumption that G is simple. Its discrete analogue is the fact that W_I is finite if $\tilde{W}$ is irreducible. (It may be helpful to consider the "discrete" versions of all the results of this section. For example, the discrete version of "$\Omega_{alg}G$ acts freely on B_G" is "the coroot lattice $\mathrm{Hom}(S^1, T)$ acts freely on t (the foundation of $\mathscr{B}_G$)"; of course the latter assertion is trivial).

Note that we have shown that $\mathscr{S}_G/\Omega_{alg}G = G$, and in fact the orbit map $\mathscr{S}_G \to G$ is given by evaluation at $t = 1$. This can also be proved directly. It reduces to the following interesting theorem, also observed by Quillen.

(4.7) THEOREM. *Suppose* $X, Y \in \mathfrak{g}$ *and* $\exp X = \exp Y$. *Then* $\exp tX = f(e^{2\pi i t}) \exp tY$ *for some* $f \in \Omega_{alg}G$. □

It is not hard to prove this directly—for example, it is enough to prove it for $G = U(n)$. Not surprisingly, however, it is also implicit in what we have already one. First, one can reduce to the case when G is simple and simply-connected. Using (1.3), one can easily reduce further to the case $X \in \mathring{\Delta}_I$, $Y = g \cdot X$ for some $g \in G$. Then $g \in C_G \exp X = G_I$, so $g = h(1)$ with $h \in L_{alg}G \cap P_I$. Let $h = \exp tX\, g \exp -tX$; then $h \in L_{alg}G$ and $f = hh(1)^{-1}$ is the desired loop.

§ 5. THE LOOPS ON A SYMMETRIC SPACE

We assume throughout this section that G is simple and simply—connected. If σ is an involution on G with fixed group K, as usual, then K is connected and G/K is simply—connected. The notations and conventions of § 1 and § 3 remain in force.

The loop space $\Omega(G/K)$ is homotopy equivalent to the space of paths in G that start at the identity and end in K. Now consider the involution τ on ΩG given by $\tau(f)(z) = \sigma(f(\bar{z}))$. The fixed group $(\Omega G)^\tau$ is clearly homeomorphic to our space of paths, since $f \in (\Omega G)^\tau$ implies $f(-1) \in K$.