

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 34 (1988)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** QUILLEN'S THEOREM ON BUILDINGS AND THE LOOPS ON A SYMMETRIC SPACE  
**Autor:** Mitchell, Stephen A.  
**Kapitel:** §4. Quillen's Theorem for Loop Groups  
**DOI:** <https://doi.org/10.5169/seals-56592>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 08.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

and  $I$  is a proper subset of  $\tilde{S}$ , then  $\tilde{W}_I$  is finite. This is obvious since the elements of  $I$  have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take  $A_s = \tilde{G}_s$ . We have  $\tilde{G}_s \tilde{B} = \tilde{G}_{c,s} \tilde{B} = \tilde{B}$   $U_s \tilde{B} = P_s$ . In particular  $P_s / \tilde{B} = \tilde{G}_s / (\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbb{C}P^1$ , which also proves Axioms (2.20) and (2.21).  $\square$

(3.2) COROLLARY.  $\Omega_{alg} G$  is a CW-complex with cells of even dimension, indexed by  $\text{Hom}(S^1, T)$ . The Poincaré series for its integral homology is  $\sum_{\lambda \in \text{Hom}(S^1, T)} t^{2\bar{l}(\lambda)}$ , where  $\bar{l}(\lambda)$  is the minimal length accruing in  $\lambda W$ . Identifying  $\text{Hom}(S^1, T)$  with  $\tilde{W}^S$ , the closure relations on the cells are given by the Bruhat order on  $\tilde{W}^S$ .  $\square$

*Remark.* An explicit formula for  $\bar{l}(\lambda)$  is given in [16], Prop. 1.25:  $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$ .

We will also need the “Iwasawa decomposition” (see [17], [27], [29]):

(3.3) THEOREM.  $\tilde{G}_C = \Omega_{alg} G \times P$ .  $\square$

*Remark.* Note that (3.3) shows that the associated building, which we will be denoted simply by  $\mathcal{B}_G$ , is a quotient of  $L_{alg} G/T \times \Delta$ . The equivalence relation is then  $(f_1 T, X) \sim (f_2 T, X)$  if  $X \in \dot{\Delta}_I$  and  $f_1 = f_2 \bmod LG \cap P_I$ .

#### § 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. Let  $G$  be a compact Lie group. Then the inclusion  $\Omega_{alg} G \rightarrow \Omega G$  is a homotopy equivalence.

If  $G$  is simply connected, let  $\mathcal{B}_G$  denote the topological building associated to the algebraic loop group  $L_{alg} G_C$  as in § 2.

(4.2) THEOREM (Quillen).  $\Omega_{alg} G$  acts freely on  $\mathcal{B}_G$ , with orbit space  $G$ .

*Proof of (4.1).* It is easy to reduce to the case when  $G$  is simply connected. Since  $B_G$  is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that  $\Omega_{alg} G \rightarrow \Omega G$  is a weak equivalence. Since both spaces have the homotopy type of a CW-complex, the map is in fact a homotopy equivalence.  $\square$

Since  $G$  is a product of simple groups (as is  $G_{\mathbb{C}}$ ), it is very easy to reduce to the case when  $G$  is simple. For the rest of this section, then, we assume  $G$  is simple and simply-connected, of rank  $l$ .

To prove 4.2, we introduce Quillen's space of special paths  $\mathcal{S}_G$ : this is the space of all paths  $[0, 1] \rightarrow G$  of the form  $f(e^{2\pi it}) \exp tX$ , where  $f \in \Omega_{\text{alg}} G$  and  $X \in \mathfrak{g}$ .  $\mathcal{S}_G$  is topologized as a quotient of  $\Omega_{\text{alg}} G \times \mathfrak{g}$ . Note that  $L_{\text{alg}} G$  acts on  $\mathcal{S}_G$  by  $h \cdot (f \exp tX) = hf \exp tXh(1)^{-1}$ . The following key lemma, whose proof is deferred, also helps to explain the significance of the parabolic subgroups  $P_I$ .

(4.3) LEMMA. Suppose  $X \in \mathring{\Delta}_I$ , then the isotropy group of  $\exp tX$  is  $L_{\text{alg}} G \cap P_I$ .

(4.4) THEOREM (Quillen).  $\mathcal{S}_G$  is  $L_{\text{alg}} G$ -equivariantly homeomorphic to the building  $\mathcal{B}_G$ .

*Proof.* The action map  $\varphi: L_{\text{alg}} G \times \Delta \rightarrow \mathcal{S}_G$  given by

$$\varphi(f, X) = f \exp tX f(1)^{-1}$$

is surjective by Theorem 1.1. If  $\varphi(f_1, X_1) = \varphi(f_2, X_2)$ , then (evaluating at  $t=1$ )  $\exp X_1$  and  $\exp X_2$  are conjugate in  $G$ , so  $X_1 = X_2$  by Theorem 1.3. We then have  $\varphi(f_1, X) = \varphi(f_2, X)$  if and only if  $f_1 = f_2$  mod the isotropy group of  $\exp tX$ . Hence, by (4.3),  $\varphi$  factors through the desired homeomorphism  $\mathcal{B}_G \rightarrow \mathcal{S}_G$ .  $\square$

*Remark.* Here we have used the Iwasawa decomposition (3.3) to identify  $\mathcal{B}_G = (\tilde{G}_{\mathbb{C}}/\tilde{B} \times \Delta)/\sim$  with  $(L_{\text{alg}} G/T \times \Delta)/\sim$ .

(4.5) LEMMA.  $L_{\text{alg}} G \cap P_I$  is generated by  $T$  and the subgroups  $\tilde{G}_i, i \in I$ .

*Proof.* We have  $P_I = \tilde{B}W_I\tilde{B}$ . By the Steinberg lemma (2.9), each  $\tilde{B}w\tilde{B} (w \in W_I)$  has the form  $XB$ , where  $X$  is a product of the  $\tilde{G}_i$ . Since  $L_{\text{alg}} G \cap XB = XT$ , the lemma follows.  $\square$

*Proof of 4.2.* The action of  $\Omega_{\text{alg}} G$  on  $\mathcal{S}_G$  is clearly free. By (4.4), the same is true for  $\mathcal{B}_G$ . Now consider the orbit space  $\mathcal{B}_G/\Omega_{\text{alg}} G$ . Since  $\mathcal{B}_G = (L_{\text{alg}} G/T \times \Delta)/\sim = (\Omega_{\text{alg}} G \times G/T \times \Delta)/\sim$ , the orbit space is a quotient of  $G/T \times \Delta$ . The equivalence relation is given by  $(g_1 T, X) \sim (g_2 T, X)$  if  $X \in \mathring{\Delta}_I$  and  $g_2 = f g_1 p$  with  $f \in \Omega_{\text{alg}} G$ ,  $p \in P_I$ . In fact  $p \in LG \cap P_I$ . Now let  $\bar{G}_I = e(LG \cap P_I)$ , where  $e$  is evaluation at  $z = 1$ . Then  $(g_1 T, X) \sim (g_2 T, X)$  if and only if  $g_1 = g_2$  mod  $\bar{G}_I$ . For if  $g_2 = f g_1 p$  as above, then  $\bar{G}_I = e(L_{\text{alg}} G \cap P_I)$ , where  $e$  is evaluation at  $z = 1$ . Then  $(g_1 T, X) \sim (g_2 T, X)$  if and only if  $g_1 = g_2$  mod  $\bar{G}_I$ . For if  $g_2 = f g_1 p$  as above, then

$g_2 = f g_1 p(1)$ , and conversely if  $g_2 = g_1 p(1)$ , then  $g_2 = f g_1 p$ , where  $f = g_2 p^{-1} g_1^{-1}$ . But by (4.5),  $\bar{G}_I = G_I$  (see § 1). In other words, the equivalence relation here coincides with the classical relation of Theorem 1.5, which has quotient  $G$ .  $\square$

*Proof of 4.3.* Fix  $X \in \dot{\Delta}_I$ . We first show that  $L_{alg}G \cap P_I$  fixes  $\exp tX$  in  $\mathcal{S}_G$ . By (4.5) it is enough to show that each  $\tilde{G}_i (i \in I)$  fixes

$$\exp tX : f(e^{2\pi it}) \exp tX f(1)^{-1} = \exp tX.$$

If  $i \neq 0$ ,  $\tilde{G}_i = G_i$  is a subgroup of the constant loops, so  $f$  is a constant  $g \in G_i$ . The desired equation is then equivalent to  $g \cdot X = X$  (recall that  $g \cdot X = \text{Ad}(g)X$ ). But since  $i \neq 0$ ,  $\alpha_i(X) = 0$ , so this is true by definition. Now suppose  $i = 0$ , so that  $X$  lies on the outer wall:  $\alpha_0(X) = 1$ . Then  $X = \frac{1}{2} \alpha_0^* + Y$ , where  $\alpha_0^* = 2\alpha_0/\alpha_0 \cdot \alpha_0$  is the coroot of  $\alpha_0$  and  $\alpha_0(Y) = 0$ .

The equation we want can be written ( $f \in \tilde{G}_0$ ):

$$f(e^{2\pi it}) = \exp tX f(1) \exp -tX$$

Since  $f(1) \in G_0$ ,  $f(1) \cdot Y = Y$ , and our equation simplifies to

$$f(e^{2\pi it}) = \exp \left( \frac{1}{2} t \alpha_0^* \right) f(1) \exp \left( -\frac{1}{2} t \alpha_0^* \right)$$

Note this is now an equation in the path space of  $G_0$ . Identifying  $G_0$  with  $SU(2)$ , it can be written

$$\begin{pmatrix} a & be^{2\pi it} \\ ce^{-2\pi it} & d \end{pmatrix} = \begin{pmatrix} e^{\pi it} & 0 \\ 0 & e^{-\pi it} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{-\pi it} & 0 \\ 0 & e^{\pi it} \end{pmatrix}$$

Where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ . This last equation is obviously correct, and we conclude that  $L_{alg}G \cap P_I$  fixes  $\exp tX$ .

Conversely, suppose

$$f \exp tX f(1)^{-1} = \exp tX, \quad \text{or} \quad f = \exp tX f(1) \exp (-tX).$$

Then  $f(1) \in C_G \exp X = G_I$ , and hence  $f(1) = h(1)$  for some  $h \in L_{alg}G \cap P_I$ . But then  $h = \exp tX h(1) \exp -tX = f$ .

A useful fact that follows from all this is:

(4.6) THEOREM. *Evaluation at 1 induces an isomorphism  $L_{alg}G \cap P_I \cong G_I$ . In particular,  $L_{alg}G \cap P_I$  is a compact Lie group.*

*Proof.* We have seen that  $e$  maps  $L_{alg}G \cap P_I$  onto  $G_I$ . The kernel is  $\Omega_{alg}G \cap P_I$ . But  $\Omega_{alg}G$  acts freely on  $\mathcal{S}_G$ , and  $L_{alg}G \cap P_I$  fixes  $\Delta_I$ , so  $\Omega_{alg}G \cap P_I = \{1\}$ .

*Remark.* As always,  $I$  is a proper subset of  $\tilde{S}$  in (4.6). Of course (4.6) also depends on our assumption that  $G$  is simple. Its discrete analogue is the fact that  $W_I$  is finite if  $\tilde{W}$  is irreducible. (It may be helpful to consider the “discrete” versions of all the results of this section. For example, the discrete version of “ $\Omega_{alg}G$  acts freely on  $B_G$ ” is “the coroot lattice  $\text{Hom}(S^1, T)$  acts freely on  $t$  (the foundation of  $\mathcal{B}_G$ )”; of course the latter assertion is trivial).

Note that we have shown that  $\mathcal{S}_G/\Omega_{alg}G = G$ , and in fact the orbit map  $\mathcal{S}_G \rightarrow G$  is given by evaluation at  $t = 1$ . This can also be proved directly. It reduces to the following interesting theorem, also observed by Quillen.

(4.7) THEOREM. *Suppose  $X, Y \in \mathfrak{g}$  and  $\exp X = \exp Y$ . Then  $\exp tX = f(e^{2\pi it}) \exp tY$  for some  $f \in \Omega_{alg}G$ .*  $\square$

It is not hard to prove this directly—for example, it is enough to prove it for  $G = U(n)$ . Not surprisingly, however, it is also implicit in what we have already one. First, one can reduce to the case when  $G$  is simple and simply-connected. Using (1.3), one can easily reduce further to the case  $X \in \dot{\Delta}_I$ ,  $Y = g \cdot X$  for some  $g \in G$ . Then  $g \in C_G \exp X = G_I$ , so  $g = h(1)$  with  $h \in L_{alg}G \cap P_I$ . Let  $h = \exp tX g \exp -tX$ ; then  $h \in L_{alg}G$  and  $f = hh(1)^{-1}$  is the desired loop.

## § 5. THE LOOPS ON A SYMMETRIC SPACE

We assume throughout this section that  $G$  is simple and simply-connected. If  $\sigma$  is an involution on  $G$  with fixed group  $K$ , as usual, then  $K$  is connected and  $G/K$  is simply-connected. The notations and conventions of § 1 and § 3 remain in force.

The loop space  $\Omega(G/K)$  is homotopy equivalent to the space of paths in  $G$  that start at the identity and end in  $K$ . Now consider the involution  $\tau$  on  $\Omega G$  given by  $\tau(f)(z) = \sigma(f(\bar{z}))$ . The fixed group  $(\Omega G)^\tau$  is clearly homeomorphic to our space of paths, since  $f \in (\Omega G)^\tau$  implies  $f(-1) \in K$ .