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and I is a proper subset of \tilde{S} , then \tilde{W}_I is finite. This is obvious since the elements of I have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take $A_s = \tilde{G}_s$. We have $\tilde{G}_s \tilde{B} = \tilde{G}_{c,s} \tilde{B} = \tilde{B}$ $U_s \tilde{B} = P_s$. In particular $P_s / \tilde{B} = \tilde{G}_s / (\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbb{C}P^1$, which also proves Axioms (2.20) and (2.21). \square

(3.2) COROLLARY. $\Omega_{alg} G$ is a CW-complex with cells of even dimension, indexed by $\text{Hom}(S^1, T)$. The Poincaré series for its integral homology is $\sum_{\lambda \in \text{Hom}(S^1, T)} t^{2\bar{l}(\lambda)}$, where $\bar{l}(\lambda)$ is the minimal length accruing in λW . Identifying $\text{Hom}(S^1, T)$ with \tilde{W}^S , the closure relations on the cells are given by the Bruhat order on \tilde{W}^S . \square

Remark. An explicit formula for $\bar{l}(\lambda)$ is given in [16], Prop. 1.25: $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$.

We will also need the “Iwasawa decomposition” (see [17], [27], [29]):

(3.3) THEOREM. $\tilde{G}_C = \Omega_{alg} G \times P$. \square

Remark. Note that (3.3) shows that the associated building, which we will be denoted simply by \mathcal{B}_G , is a quotient of $L_{alg} G/T \times \Delta$. The equivalence relation is then $(f_1 T, X) \sim (f_2 T, X)$ if $X \in \dot{\Delta}_I$ and $f_1 = f_2 \bmod LG \cap P_I$.

§ 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. Let G be a compact Lie group. Then the inclusion $\Omega_{alg} G \rightarrow \Omega G$ is a homotopy equivalence.

If G is simply connected, let \mathcal{B}_G denote the topological building associated to the algebraic loop group $L_{alg} G_C$ as in § 2.

(4.2) THEOREM (Quillen). $\Omega_{alg} G$ acts freely on \mathcal{B}_G , with orbit space G .

Proof of (4.1). It is easy to reduce to the case when G is simply connected. Since B_G is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that $\Omega_{alg} G \rightarrow \Omega G$ is a weak equivalence. Since both spaces have the homotopy type of a CW-complex, the map is in fact a homotopy equivalence. \square

Since G is a product of simple groups (as is G_c), it is very easy to reduce to the case when G is simple. For the rest of this section, then, we assume G is simple and simply-connected, of rank l .

To prove 4.2, we introduce Quillen's space of special paths \mathcal{S}_G : this is the space of all paths $[0, 1] \rightarrow G$ of the form $f(e^{2\pi it}) \exp tX$, where $f \in \Omega_{alg}G$ and $X \in \mathfrak{g}$. \mathcal{S}_G is topologized as a quotient of $\Omega_{alg}G \times \mathfrak{g}$. Note that $L_{alg}G$ acts on \mathcal{S}_G by $h \cdot (f \exp tX) = hf \exp tXh(1)^{-1}$. The following key lemma, whose proof is deferred, also helps to explain the significance of the parabolic subgroups P_I .

(4.3) LEMMA. Suppose $X \in \mathring{\Delta}_I$, then the isotropy group of $\exp tX$ is $L_{alg}G \cap P_I$.

(4.4) THEOREM (Quillen). \mathcal{S}_G is $L_{alg}G$ -equivariantly homeomorphic to the building \mathcal{B}_G .

Proof. The action map $\varphi: L_{alg}G \times \Delta \rightarrow \mathcal{S}_G$ given by

$$\varphi(f, X) = f \exp tX f(1)^{-1}$$

is surjective by Theorem 1.1. If $\varphi(f_1, X_1) = \varphi(f_2, X_2)$, then (evaluating at $t=1$) $\exp X_1$ and $\exp X_2$ are conjugate in G , so $X_1 = X_2$ by Theorem 1.3. We then have $\varphi(f_1, X) = \varphi(f_2, X)$ if and only if $f_1 = f_2$ mod the isotropy group of $\exp tX$. Hence, by (4.3), φ factors through the desired homeomorphism $\mathcal{B}_G \rightarrow \mathcal{S}_G$. \square

Remark. Here we have used the Iwasawa decomposition (3.3) to identify $\mathcal{B}_G = (\tilde{G}_c/\tilde{B} \times \Delta)/\sim$ with $(L_{alg}G/T \times \Delta)/\sim$.

(4.5) LEMMA. $L_{alg}G \cap P_I$ is generated by T and the subgroups $\tilde{G}_i, i \in I$.

Proof. We have $P_I = \tilde{B}W_I\tilde{B}$. By the Steinberg lemma (2.9), each $\tilde{B}w\tilde{B}$ ($w \in W_I$) has the form XB , where X is a product of the \tilde{G}_i . Since $L_{alg}G \cap XB = XT$, the lemma follows. \square

Proof of 4.2. The action of $\Omega_{alg}G$ on \mathcal{S}_G is clearly free. By (4.4), the same is true for \mathcal{B}_G . Now consider the orbit space $\mathcal{B}_G/\Omega_{alg}G$. Since $\mathcal{B}_G = (L_{alg}G/T \times \Delta)/\sim = (\Omega_{alg}G \times G/T \times \Delta)/\sim$, the orbit space is a quotient of $G/T \times \Delta$. The equivalence relation is given by $(g_1T, X) \sim (g_2T, X)$ if $X \in \mathring{\Delta}_I$ and $g_2 = fg_1p$ with $f \in \Omega_{alg}G$, $p \in P_I$. In fact $p \in LG \cap P_I$. Now let $\bar{G}_I = e(LG \cap P_I)$, where e is evaluation at $z = 1$. Then $(g_1T, X) \sim (g_2T, X)$ if and only if $g_1 = g_2$ mod \bar{G}_I . For if $g_2 = fg_1p$ as above, then $\bar{G}_I = e(L_{alg}G \cap P_I)$, where e is evaluation at $z = 1$. Then $(g_1T, X) \sim (g_2T, X)$ if and only if $g_1 = g_2$ mod \bar{G}_I . For if $g_2 = fg_1p$ as above, then

$g_2 = f g_1 p(1)$, and conversely if $g_2 = g_1 p(1)$, then $g_2 = f g_1 p$, where $f = g_2 p^{-1} g_1^{-1}$. But by (4.5), $\bar{G}_I = G_I$ (see § 1). In other words, the equivalence relation here coincides with the classical relation of Theorem 1.5, which has quotient G . \square

Proof of 4.3. Fix $X \in \dot{\Delta}_I$. We first show that $L_{alg}G \cap P_I$ fixes $\exp tX$ in \mathcal{S}_G . By (4.5) it is enough to show that each $\tilde{G}_i (i \in I)$ fixes

$$\exp tX : f(e^{2\pi it}) \exp tX f(1)^{-1} = \exp tX.$$

If $i \neq 0$, $\tilde{G}_i = G_i$ is a subgroup of the constant loops, so f is a constant $g \in G_i$. The desired equation is then equivalent to $g \cdot X = X$ (recall that $g \cdot X = \text{Ad}(g)X$). But since $i \neq 0$, $\alpha_i(X) = 0$, so this is true by definition. Now suppose $i = 0$, so that X lies on the outer wall: $\alpha_0(X) = 1$. Then $X = \frac{1}{2} \alpha_0^* + Y$, where $\alpha_0^* = 2\alpha_0/\alpha_0 \cdot \alpha_0$ is the coroot of α_0 and $\alpha_0(Y) = 0$.

The equation we want can be written ($f \in \tilde{G}_0$):

$$f(e^{2\pi it}) = \exp tX f(1) \exp -tX$$

Since $f(1) \in G_0$, $f(1) \cdot Y = Y$, and our equation simplifies to

$$f(e^{2\pi it}) = \exp \left(\frac{1}{2} t \alpha_0^* \right) f(1) \exp \left(-\frac{1}{2} t \alpha_0^* \right)$$

Note this is now an equation in the path space of G_0 . Identifying G_0 with $SU(2)$, it can be written

$$\begin{pmatrix} a & be^{2\pi it} \\ ce^{-2\pi it} & d \end{pmatrix} = \begin{pmatrix} e^{\pi it} & 0 \\ 0 & e^{-\pi it} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{-\pi it} & 0 \\ 0 & e^{\pi it} \end{pmatrix}$$

Where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$. This last equation is obviously correct, and we conclude that $L_{alg}G \cap P_I$ fixes $\exp tX$.

Conversely, suppose

$$f \exp tX f(1)^{-1} = \exp tX, \quad \text{or} \quad f = \exp tX f(1) \exp (-tX).$$

Then $f(1) \in C_G \exp X = G_I$, and hence $f(1) = h(1)$ for some $h \in L_{alg}G \cap P_I$. But then $h = \exp tX h(1) \exp -tX = f$.

A useful fact that follows from all this is:

(4.6) THEOREM. *Evaluation at 1 induces an isomorphism $L_{alg}G \cap P_I \cong G_I$. In particular, $L_{alg}G \cap P_I$ is a compact Lie group.*

Proof. We have seen that e maps $L_{alg}G \cap P_I$ onto G_I . The kernel is $\Omega_{alg}G \cap P_I$. But $\Omega_{alg}G$ acts freely on \mathcal{S}_G , and $L_{alg}G \cap P_I$ fixes Δ_I , so $\Omega_{alg}G \cap P_I = \{1\}$.

Remark. As always, I is a proper subset of \tilde{S} in (4.6). Of course (4.6) also depends on our assumption that G is simple. Its discrete analogue is the fact that W_I is finite if \tilde{W} is irreducible. (It may be helpful to consider the “discrete” versions of all the results of this section. For example, the discrete version of “ $\Omega_{alg}G$ acts freely on B_G ” is “the coroot lattice $\text{Hom}(S^1, T)$ acts freely on t (the foundation of \mathcal{B}_G)”; of course the latter assertion is trivial).

Note that we have shown that $\mathcal{S}_G/\Omega_{alg}G = G$, and in fact the orbit map $\mathcal{S}_G \rightarrow G$ is given by evaluation at $t = 1$. This can also be proved directly. It reduces to the following interesting theorem, also observed by Quillen.

(4.7) THEOREM. *Suppose $X, Y \in \mathfrak{g}$ and $\exp X = \exp Y$. Then $\exp tX = f(e^{2\pi it}) \exp tY$ for some $f \in \Omega_{alg}G$.* \square

It is not hard to prove this directly—for example, it is enough to prove it for $G = U(n)$. Not surprisingly, however, it is also implicit in what we have already one. First, one can reduce to the case when G is simple and simply-connected. Using (1.3), one can easily reduce further to the case $X \in \dot{\Delta}_I$, $Y = g \cdot X$ for some $g \in G$. Then $g \in C_G \exp X = G_I$, so $g = h(1)$ with $h \in L_{alg}G \cap P_I$. Let $h = \exp tX g \exp -tX$; then $h \in L_{alg}G$ and $f = hh(1)^{-1}$ is the desired loop.

§ 5. THE LOOPS ON A SYMMETRIC SPACE

We assume throughout this section that G is simple and simply-connected. If σ is an involution on G with fixed group K , as usual, then K is connected and G/K is simply-connected. The notations and conventions of § 1 and § 3 remain in force.

The loop space $\Omega(G/K)$ is homotopy equivalent to the space of paths in G that start at the identity and end in K . Now consider the involution τ on ΩG given by $\tau(f)(z) = \sigma(f(\bar{z}))$. The fixed group $(\Omega G)^\tau$ is clearly homeomorphic to our space of paths, since $f \in (\Omega G)^\tau$ implies $f(-1) \in K$.