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x is a *cut point* (with respect to p) if there is a geodesic from p to x that minimizes arc length up to x but no further. The *cut locus* is the set of cut points. Similarly a vector X in the tangent space T_p is a *tangent cut point* if $\exp_p X$ is a cut point along the geodesic $\exp_p(tX)$. The *tangent cut locus* is the set of all such points in T_p , and is homeomorphic to the unit sphere in T_p . When $M = G/K$ we take $p = 1$.

(2.26) THEOREM. *Let G/K be a simply-connected symmetric space, with G simple. Then the tangent cut locus is precisely the K -orbit in \mathfrak{m} of the outer wall of the Cartan simplex $\Delta_{\mathfrak{m}}$. It is therefore canonically identified with the topological building of the associated real form $G_{\mathbf{R}}$.*

As usual, the assumption G simple is just for convenience. We sketch the proof: the first assertion is a fairly easy consequence of Theorem (1.8), and is left to the reader. Now consider the building $\mathcal{B}_{G_{\mathbf{R}}}$. It is a quotient space of $G_{\mathbf{R}}/B_{\mathbf{R}} \times \Delta_0 = K/C_K t_{\mathfrak{m}} \times \Delta_0$, where Δ_0 is a simplex of dimension $(\text{rank } G/K) - 1$; we take Δ_0 to be the outer wall of $\Delta_{\mathfrak{m}}$. For each $I \leq S_{G/K}$, let Δ_I temporarily denote the corresponding face of Δ_0 ; i.e. $\{X \in \Delta_0 : \alpha_i(X) = 0 \ \forall i \in I\}$. Then the K -orbit of Δ_0 in \mathfrak{m} , $K\Delta_0$, is also a quotient of $K/C_K t_{\mathfrak{m}} \times \Delta_0$. The relations are $(k_1 X) \sim (k_2 X)$ if $X \in \Delta_I$ and $k_1 = k_2 \text{ mod } K_I$. But $K_I = K \cap \mathcal{O}_I$, so these relations are identical to the ones that define the building. \square

§ 3. LOOP GROUPS

Let $LG, LG_{\mathbf{C}}$ denote the free loop spaces. Let $G_{\mathbf{C}}$ denote the group of loops which are restrictions of regular maps $\mathbf{C}^* \rightarrow G_{\mathbf{C}}$, and let $L_{\text{alg}}G = L_{\text{alg}}G_{\mathbf{C}} \cap LG$. Thus if we fix an embedding $G_{\mathbf{C}} \subset GL(n, \mathbf{C})$, $L_{\text{alg}}G$ consists of the loops f in LG admitting a finite Laurent expansion $f(z) = \sum_{k=-m}^m A_k z^k$, whereas $L_{\text{alg}}G_{\mathbf{C}}$ consists of the loops f in $LG_{\mathbf{C}}$ such that both f and f^{-1} admit finite Laurent expansions. We will also write $\tilde{G}_{\mathbf{C}}$ for $L_{\text{alg}}G_{\mathbf{C}}$. In fact $\tilde{G}_{\mathbf{C}}$ is the group of points over $\mathbf{C}[z, z^{-1}]$ of the algebraic group $G_{\mathbf{C}}$. Its Lie algebra is the loop algebra $\tilde{g}_{\mathbf{C}}$ of regular maps $\mathbf{C}^* \rightarrow g_{\mathbf{C}}$. The integer m in the above Laurent expansion defines a filtration of $\tilde{G}_{\mathbf{C}}$ by finite dimensional subspaces; we give $\tilde{G}_{\mathbf{C}}$ the corresponding weak topology.

Let P denote the subgroup of $\tilde{G}_{\mathbf{C}}$ consisting of regular maps $\mathbf{C} \rightarrow G_{\mathbf{C}}$ (i.e. maps with nonnegative Laurent expansion, or $G_{\mathbf{C}[z]}$), and let \tilde{B} denote the Iwahori subgroup: $\{f \in P : f(0) \in B^{-}\}$. Finally, let $\tilde{N} = L_{\text{alg}}N_{\mathbf{C}}$, and recall that \tilde{W} can be regarded as a "subgroup" of $\tilde{G}_{\mathbf{C}}$, since $R \leq \text{Hom}(S^1, T) \leq L_{\text{alg}}T$. More precisely, we have $\tilde{N}/T_{\mathbf{C}} = \hat{W}$, and $\tilde{W} \subset \hat{W}$.

The *affine root system* Φ is the set $\mathbf{Z} \times \Phi$. It can be thought of as a set of affine linear functionals on t , but for our purposes it is just a device for encoding combinatorial information about the affine Weyl group and \tilde{G}_C . In particular, to each $(n, \alpha) \in \Phi$ we associate a root subalgebra $X_{n, \alpha}$ of \tilde{g}_C consisting of the regular maps $\mathbf{C}^* \rightarrow X_\alpha$ homogeneous of degree n . These subalgebras are one-dimensional, and are precisely the nontrivial eigenspaces of the following T^{l+1} action: The constant loops T^l act in the obvious way, and the extra S^1 factor acts by rotating the loops. We also have root subgroups $U_{(n, \alpha)} = \exp X_{n, \alpha} \leq \tilde{G}_C$. One can easily check that \tilde{W} (acting by left conjugation) permutes the root subgroups. The resulting action of \tilde{W} on $\tilde{\Phi}$ is given by $(w\lambda) \cdot (n, \alpha) = (n + \alpha(\lambda), w\alpha)$ for $\lambda \in \text{hom}(S^1, T)$, $w \in W$. The various additional structures associated with ordinary root systems can be defined here as well. The positive roots $\tilde{\Phi}^+$ are the (n, α) with $n \geq 1$ or $n = 0$ and $\alpha < 0$ (note these correspond to the Iwahori subgroup \tilde{B}); the remaining roots are negative. As in the finite case, the length of an element σ in \tilde{W} is equal to the number of positive roots taken to negative roots by σ (in particular this latter number is finite, as is clear anyway from the above formula for the \tilde{W} action). The simple affine roots are defined as the set of elements of $\tilde{\Phi}^+$ which are indecomposable with respect to addition: $(m, \alpha) + (n, \beta) = (m+n, \alpha+\beta)$ (if $\alpha+\beta$ is a root). Hence the simple roots are $(0, -\alpha), \dots, (0, -\alpha_l)$ and $(1, \alpha_0)$.

To each root (n, α) , we can also associate a "little SL_2 " subgroup generated by $U_{n, \alpha}$ and $U_{-n, -\alpha}$. In particular $\tilde{G}_{C, i}$ is the subgroup corresponding to the i th simple affine root, $0 \leq i \leq l$. Thus $\tilde{G}_{C, i} = G_{C, i}$ if $i \neq 0$, and $\tilde{G}_{C, 0}$ corresponds to $(1, \alpha_0)$. For example, if $G = SU(2)$, $\tilde{G}_{C, 0}$ is the subgroup of matrices $\begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix}$ with $ad - bc = 1$. We let $\tilde{G}_i = \tilde{G}_{C, i} \cap LG$. Again $\tilde{G}_i = G_i$ if $i \neq 0$. Note that for all i , evaluation at $z = 1$ gives an isomorphism $\tilde{G}_i \xrightarrow{\cong} G_i \cong SU(2)$.

(3.1) THEOREM. Assume G is simply-connected. Then $(\tilde{G}_C, \tilde{B}, \tilde{N}, \tilde{S})$ is a topological Tits system satisfying the four axioms of § 2.

Proof. That $(\tilde{G}_C, \tilde{B}, \tilde{N}, \tilde{S})$ is a Tits system in the ordinary sense is essentially due to Iwahori and Matsumoto [16]. (They work over a complete local field K ; here we take K to be the field of infinite Laurent series bounded below. It is not hard to get from the Chevalley group G_K to $G_{\mathbf{C}[z, z^{-1}]} = \tilde{G}_C$.) See also Kac and Peterson [17].

Clearly \tilde{B} and \tilde{N} are closed subgroups and \tilde{W} is discrete. For Axiom (2.11) we need to show that if \tilde{W} is an irreducible affine Weyl group,

and I is a proper subset of \tilde{S} , then \tilde{W}_I is finite. This is obvious since the elements of I have a common fixed point (i.e. the intersection of the corresponding reflection hyperplanes is nonempty). In Axiom (2.12) we take $A_s = \tilde{G}_s$. We have $\tilde{G}_s \tilde{B} = \tilde{G}_{c,s} \tilde{B} = \tilde{B}$ $U_s \tilde{B} = P_s$. In particular $P_s / \tilde{B} = \tilde{G}_s / (\tilde{G}_s \cap \tilde{B}) \cong SU(2)/T = \mathbb{C}P^1$, which also proves Axioms (2.20) and (2.21). \square

(3.2) COROLLARY. $\Omega_{alg} G$ is a CW-complex with cells of even dimension, indexed by $\text{Hom}(S^1, T)$. The Poincaré series for its integral homology is $\sum_{\lambda \in \text{Hom}(S^1, T)} t^{2\bar{l}(\lambda)}$, where $\bar{l}(\lambda)$ is the minimal length accruing in λW . Identifying $\text{Hom}(S^1, T)$ with \tilde{W}^S , the closure relations on the cells are given by the Bruhat order on \tilde{W}^S . \square

Remark. An explicit formula for $\bar{l}(\lambda)$ is given in [16], Prop. 1.25: $\bar{l}(\lambda) = (\sum_{\alpha > 0} |\alpha(\lambda)|) - |\{\alpha > 0 : \alpha(\lambda) > 0\}|$.

We will also need the “Iwasawa decomposition” (see [17], [27], [29]):

(3.3) THEOREM. $\tilde{G}_C = \Omega_{alg} G \times P$. \square

Remark. Note that (3.3) shows that the associated building, which we will be denoted simply by \mathcal{B}_G , is a quotient of $L_{alg} G/T \times \Delta$. The equivalence relation is then $(f_1 T, X) \sim (f_2 T, X)$ if $X \in \Delta_I$ and $f_1 = f_2 \bmod LG \cap P_I$.

§ 4. QUILLEN'S THEOREM FOR LOOP GROUPS

In this section we will give Quillen's proof of the following theorem.

(4.1) THEOREM. Let G be a compact Lie group. Then the inclusion $\Omega_{alg} G \rightarrow \Omega G$ is a homotopy equivalence.

If G is simply connected, let \mathcal{B}_G denote the topological building associated to the algebraic loop group $L_{alg} G_C$ as in § 2.

(4.2) THEOREM (Quillen). $\Omega_{alg} G$ acts freely on \mathcal{B}_G , with orbit space G .

Proof of (4.1). It is easy to reduce to the case when G is simply connected. Since B_G is contractible by Theorem 2.16, we conclude at once from Theorem (4.2) that $\Omega_{alg} G \rightarrow \Omega G$ is a weak equivalence. Since both spaces have the homotopy type of a CW-complex, the map is in fact a homotopy equivalence. \square