Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 34 (1988)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: QUILLEN'S THEOREM ON BUILDINGS AND THE LOOPS ON A SYMMETRIC SPACE

Autor: Mitchell, Stephen A.

Kapitel: §1. Notation and Preliminaries

DOI: https://doi.org/10.5169/seals-56592

analogue for symmetric spaces, but see [9] for an example); there is a very thorough account of this approach in [29].

I would like to thank Suren Fernando for some very helpful conversations.

§ 1. Notation and Preliminaries

Except in § 2, G will always denote a compact connected Lie group of rank l; usually we will assume also that G is simple and simply-connected. Fix once and for all a maximal torus T in G, and let N denote the normalizer $N_G T$. The Weyl group W is N/T. Lie algebras are denoted as usual by Gothic letter: $\mathfrak{g}$, $\mathfrak{t}$, etc. To each G we can associate a reductive complex algebraic group $G_{\mathbf{C}}$—the complexification of G—with Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$. It contains G as a maximal compact subgroup, and as the fixed group of an anti-complex involution. In fact $G \to G_{\mathbf{C}}$ defines an equivalence of categories (compact Lie groups) $\leftrightarrow$ (reductive complex algebraic groups).

$G_{\mathbf{C}}$ has a Borel subgroup (maximal connected solvable subgroup) B, unique up to conjugacy, which we can assume contains the Cartan subgroup (maximal algebraic torus) $T_{\mathbf{C}}$. There is a split extension $U \to B \to T_{\mathbf{C}}$ where U is the unipotent radical of B. There is also an opposite Borel subgroup B^- such that $B \cap B^- = T_{\mathbf{C}}$; it fits into a similar split extension $U^- \to B^- \to T_{\mathbf{C}}$. On the Lie algebra level we have $\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{u} \oplus \mathfrak{u}^-$, with $\mathfrak{u} \oplus \mathfrak{u}^-$ being precisely the sum of the nontrivial eigenspaces for the adjoint action of $\mathfrak{t}_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$. The corresponding eigenfunctions $\lambda : \mathfrak{t}_{\mathbf{C}} \to \mathbf{C}$ map $\mathfrak{t}$ into $i\mathbf{R}$; as is customary we replace each λ by $\alpha = \lambda/2\pi i$ to obtain a set Φ of nontrivial $\mathbf{R}$-valued linear functionals on $\mathfrak{t}$-the real roots. These form a (reduced, crystallographic) root system in $\mathfrak{t}^*$. The positive roots Φ^+ correspond to $\mathfrak{u}$, the negative roots Φ^- to $\mathfrak{u}^-$. A simple system of roots $\alpha_1, ..., \alpha_l$ (here we assume G is semisimple of rank l) is then uniquely determined as the set of positive roots which are not decomposable as sums of positive roots. If we assume G is simple, so that Φ is irreducible, there is a unique "highest root" α_0, which is characterized by the property that for every positive root α, $\alpha_0 + \alpha$ is not a root. The corresponding eigenspace in $\mathfrak{u}$ is precisely the center of $\mathfrak{u}$. And, speaking of eigenspaces, let X_α denote the eigenspace (or "root subalgebra") of $\mathfrak{g}_{\mathbf{C}}$ associated to $\alpha \in \Phi$. For each α, the subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by X_α and $X_{-\alpha}$ is isomorphic to $sl(2, \mathbf{C})$. The corresponding subgroup, isomorphic to $SL_2\mathbf{C}$ or $PSL_2\mathbf{C}$, is $G_{\mathbf{C},\alpha}$. Choosing generators E_α for the X_α, we obtain a basis for $\mathfrak{g}_{\mathbf{C}}$, consisting of the $E_\alpha(\alpha \in \Phi)$ and $H_\alpha = [E_\alpha, E_{-\alpha}]$ $(\alpha \in \Phi^+)$.

The basis above can be chosen so that the antilinear map $\mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ defined by $E_\alpha \to -E_{-\alpha}$ is a Lie algebra automorphism with fixed algebra $\mathfrak{g}$. In particular, then, we have $\mathfrak{g} = \mathfrak{t} \oplus (\oplus_{\alpha \in \Phi^+} Y_\alpha)$, where Y_α is spanned by $E_\alpha - E_{-\alpha}$ and $i(E_\alpha + E_{-\alpha})$. The Y_α are "eigenspaces" for the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$. Each Y_α generates a Lie algebra isomorphic to $\mathfrak{su}(2)$. The corresponding subgroups G_α, isomorphic to $SU(2)$ or $SO(3)$, are extremely important; for example, they generate G (if G is semisimple). Note G_α is a maximal compact subgroup of $G_{\mathbf{C},\alpha}$.

In $\mathfrak{t}$ there are three lattices: the coroot lattice R, spanned by the coroots $\alpha^\vee = 2\alpha/\alpha\cdot\alpha$ ($\mathfrak{t}$ is identified with $\mathfrak{t}^*$ via a W-invariant inner product), the integral lattice $I = Ker(\exp\colon \mathfrak{t} \to T)$, and the coweight lattice $J = \{X \in \mathfrak{t}\colon \alpha(x) \in \mathbf{Z}\ \forall\ \alpha \in \Phi\}$. We have $R \leqslant I \leqslant J$, with $I/R \cong \pi_1 G$ and $J/I \cong C(G)$. If we think of R as a group of isometries (translation) of $\mathfrak{t}$, then R is normalized by W; the *affine Weyl group* $\tilde{W}$ is the semidirect product RW. Next, consider the Stiefel diagram, which consists of the hyperplanes $P_{\alpha,n} = \{X \in t\colon \alpha(x) = n\}$ ($\alpha \in \Phi$, $n \in \mathbf{Z}$). The connected components of the complement of the diagram are the *alcoves*, and we have:

(1.1) THEOREM. *(a) $\tilde{W}$ acts simply transitively on the alcoves; (b) $\tilde{W}$ is generated by the reflections in the walls of any fixed alcove.* □

Now let $\mathscr{C}^+$ be the positive Weyl chamber: $\{X \in t\colon \alpha(x) > 0\ \forall\ \alpha \in \Phi^+\}$. Assume (for convenience) that G is simple. Then as our standard alcove we take $\mathscr{A}^+ = \{X \in \mathscr{C}^+ : \alpha_0(X) < 1\}$. The closure Δ of $\mathscr{A}^+$ is an l-simplex—the *Cartan simplex*; its walls are the hyperplanes $\alpha_i = 0 (1 \leqslant i \leqslant l)$, $\alpha_0 = 1$. The wall $\alpha_0 = 1$ will be called the *outer wall.* Thus $\tilde{W}$ is generated by the set $\tilde{S} = S \cup \{s_0\}$, where s_0 is reflection in the outer wall. For each subset I of $\tilde{S}$ the I-face Δ_I of Δ is defined by $\Delta_I = \{X \in \Delta\colon \alpha_i(x) = 0$ if $i \in I$, $i \neq 0$, $\alpha_0(x) = 1$ if $0 \in I\}$. (Here $\tilde{S} = \{s_0, \cdots, s_l\} \equiv \{0, 1, \cdots l\}$). We let $\mathring{\Delta}_I$ denote the interior of Δ_I, so that Δ is the disjoint union of the $\mathring{\Delta}_I$. The isotropy group in $\tilde{W}$ of any $X \in \mathring{\Delta}_I$ is precisely $\tilde{W}_I$ (the subgroup generated by I).

(1.2) THEOREM. *Suppose $X, Y \in \Delta$ and $\sigma X = Y$ for some $\sigma \in \tilde{W}$. Then $X = Y$ and $\sigma \in \tilde{W}_I$, where $I = \{s \in \tilde{S}\colon sX = X\}$.* □

The most important feature of Δ, for our purposes, is the following:

(1.3) THEOREM. *Every element of G is conjugate to $\exp X$ for some $X \in \Delta$. If G is simply-connected, X is unique.*

[The proof of this classical theorem is easily obtained from what we have stated so far, together with the conjugacy of maximal tori and the fact that two elements of T conjugate in G are conjugate by an element of W]. □

The first part of (1.3) asserts that the map $G/T \times \Delta \xrightarrow{\pi} G$ given by $\pi(gT, X) = g \exp X g^{-1}$ is surjective. Thus G is a quotient space $G/T \times \Delta/\sim$ for a certain equivalence relation $\sim$. If G is simply-connected, the second part asserts that the equivalence relation is given by $(g_1T, X_1) \sim (g_2T, X_2)$ if and only if $X_1 = X_2 = X$ (say), and $g_1 = g_2 \bmod C_G \exp X$. Now $C_g \exp X(\{Y \in g : (\exp X) \cdot Y = Y\})$ is easily determined (we write $g \cdot X$ for $(\mathrm{Ad}g)(X)$): $C_g \exp X = (\bigoplus_{\alpha(x) \in \mathbf{Z}} V_\alpha) \oplus t$, and furthermore $\{\alpha \in \Phi : \alpha(X) \in \mathbf{Z}\}$ is generated by the simple roots it contains—provided that $(-\alpha_0)$ is counted as a simple root. (Of course for $X \in \Delta$, $\alpha(x) \in \mathbf{Z}$ means $\alpha(x) = 0, \pm 1$). In other words, if $X \in \mathring{\Delta}_I$, the identity component of $C_G \exp X$ is the (closed) subgroup G_I generated by T and the G_{α_i}, $i \in I$. We recall here that although centralizers of tori are always connected, centralizers of elements need not be. Fortunately, however, there is the following result.

(1.4) THEOREM (Borel [2], Bott [unpublished]). *If Θ is an automorphism of a simply-connected compact Lie group G, the fixed group of Θ is connected.* □

In particular centralizers are connected in this case, so $C_G \exp X = G_I$. We summarize the preceeding discussion in the next theorem.

(1.5) THEOREM. *Let G be a simple, simply-connected compact Lie group, regarded as a quotient space of $G/T \times \Delta$ as above. Then the equivalence relation on $G/T \times \Delta$ is given by $(g_1T, X) \sim (g_2T, X)$ if $X \in \mathring{\Delta}_I$ and $g_1 = g_2 \bmod G_I$.* □

We turn next to symmetric spaces. Let σ be an involution of a semi-simple G with fixed group K, and let K' be any subgroup of K containing the identity component. For our purposes a symmetric space is by definition a space of the form G/K'. However we will consider exclusively simply-connected symmetric spaces; in that case K' is necessarily connected. Lifting σ to an involution $\tilde{\sigma}$ of the universal cover $\tilde{G}$ of G, we see that $G/K' = \tilde{G}/K''$, where K'' is the fixed group of $\tilde{\sigma}$. Hence we may assume without loss of generality that G itself is simply-connected, and in that case the Borel-Bott theorem guarantees that K is connected. The induced involution on $\mathfrak{g}$ will also be denoted by σ. We have $\mathfrak{g} = k \oplus m$, where $\mathfrak{m}$ is the (-1)-eigenspace of σ. Let $M = \exp m$. Then:

(1.6) THEOREM. *The map* $\eta: G/K \to M$ *given by* $\eta(gk) = g\sigma(g^{-1})$ *is a* K*-equivariant homeomorphism. (*K *acts on* M *by conjugation.)* □

From now on we identify G/K with M. Let $\mathfrak{t}_{\mathfrak{m}}$ be a maximal abelian subspace of $\mathfrak{m}$ (any two such are K-conjugate); we can assume $\mathfrak{t}_{\mathfrak{m}} \subset t$. The torus $T_m = \exp \mathfrak{t}_m$ is a *maximal torus* of M (or of G/K). The *relative Weyl group* $W_{G,K}$ is $N_K t_m / C_K t_m$; as in the absolute case, it is a finite group.

Now the involution σ on $\mathfrak{g}$ (resp. G) extends uniquely to an anti-complex involution on $g_{\mathbf{C}}$ (resp. $G_{\mathbf{C}}$). Passing to fixed points, we obtain the associated real forms $G_{\mathbf{R}} = (G_{\mathbf{C}})^{\sigma}$ (not to be confused with $(G^{\sigma})_{\mathbf{C}}$!) and $\mathfrak{g}_{\mathbf{R}} = (\mathfrak{g}_{\mathbf{C}})^{\sigma}$. $G_{\mathbf{R}}$ is semisimple real Lie group, containing K as a maximal compact, and will play an important role.

Up to conjugacy, we can assume that σ is in "normal form": σ preserves $\mathfrak{t}_{\mathbf{C}}$, and commutes with the "compact" involution of $\mathfrak{g}_{\mathbf{C}}$ (the involution with fixed algebra $\mathfrak{g}$). With this assumption, we now consider the associated *relative root system*. Since σ is antilinear, its action on $\mathfrak{t}_{\mathbf{C}}^*$ is given by $(\sigma\lambda)(x) = \overline{\lambda(\sigma x)}$. This action permutes the complex roots, and yields an involution on the real roots $\Phi: (\sigma\alpha)(x) = -\alpha(\sigma x)$. Let Φ_0 denote the set of roots which restrict to zero on $\mathfrak{t}_m$; and let W_0 denote the associated Weyl group (note Φ_0 is spanned by the simple roots it contains; W_0 is the subgroup generated by the corresponding simple reflections). The *relative root system* Σ is the set of nonzero linear functionals β on $\mathfrak{t}_m$ which are restrictions of roots $\alpha \in \Phi$. One can show that Σ is indeed a root system, although it is not necessarily reduced—i.e., there may be roots β such that 2β is also a root. The following result is due to Satake [31]:

(1.7) THEOREM. *There is a base* B *(simple system of roots) for* Φ *such that if* Φ^+ *is the corresponding set of positive roots,* σ *preserves* $\Phi^+ - \Phi_0$. *Furthermore any such base satisfies (a)* $B \cap \Phi_0$ *is a base for* Φ_0 *and (b) For each* $\alpha \in B - \Phi_0$, *there is a unique* $\alpha' \in B - \Phi_0$ *such that* $\sigma\alpha = \alpha' \bmod \mathbf{Z}\Phi$. □

Using this theorem, the *Satake diagram* of G/K can be described as follows. Start with the Dynkin diagram of G; its nodes are labelled by the simple roots of Φ (or by the set S). Color the nodes belonging to Φ_0 black and color the remaining nodes white. By part (b) there is an involution (possibly trivial) on the set of white nodes; this is indicated by drawing double arrows $\leftrightarrow$ between the nodes of each nontrivial orbit. Six examples are given in § 6; see [13], pp. 532-4 for a list of all possible Satake diagrams. To capture all of the structure of G/K another diagram is needed, which we will call the Dynkin Diagram of G/K. First define the *multiplicity* m_β of a root β in Σ to be the number of roots in Φ which restrict to β. Then the Dynkin diagram of G/K is the Dynkin diagram of Σ with the nodes labelled by their multiplicities; if β is a simple root such that 2β is also a root, the β-node is to be labelled by $(m_\beta, m_{2\beta})$. Again, see § 6 for examples; for the moment we just mention an extreme case: If G/K has maximal rank—i.e. $\mathfrak{t}_m = \mathfrak{t}$—then $G_{\mathbf{R}}$ is the so-called split real form of $G_{\mathbf{C}}$. The nodes of the Satake diagram are then all white, with trivial involution, $\Phi = \Sigma$ and $m_\alpha = 1$ for all α. For example, take $G = SU(n)$, $\sigma(A) = \bar{A}$, $K = SO(n)$ and $G_{\mathbf{R}} = SL(n, \mathbf{R})$. (The opposite extreme —all nodes on the Satake diagram black—corresponds to the compact involution on $G_{\mathbf{C}}$ (*so* $\sigma \mid_G = 1$), and will be ignored.)

For our purposes it is necessary to consider the extended Satake and Dynkin diagrams. We recall here that the extended Dynkin diagram of an irreducible (reduced) root system is obtained formally by considering $-\alpha_0$ as a simple root and adjoining a corresponding node to the ordinary Dynkin diagram. (For us this definition is motivated by loop groups (§ 3), but it has many other uses—for example, in the Borel-de Siebenthal classification of maximal rank subgroups of G [3]). Now in view of (1.7) it is clear that σ_0 restricts to the highest root of Σ, and so in particular restricts non-trivially. Hence the extended Satake diagram is obtained by coloring the $(-\alpha_0)$-node white (and leaving it fixed under the involution, for reasons which should become clear later). The extended Dynkin diagram for G/K is obtained from the ordinary one by adjoining $-\alpha_0$ and labelling it by its multiplicity ($2\alpha_0$ is never a root).

Next, we will need the analogues of the subgroups $G_{\mathbf{C},\alpha}$ and G_α in the real form $G_{\mathbf{R}}$. Let β be a simple root in Σ, and let I_β be the subset of S determined as follows (cf. [22], pp. 135-36): In the Satake diagram form the subdiagram consisting of the black nodes and the set of white nodes that correspond to β under restriction (there are either one or two such white nodes). Then, in this subdiagram, take the path component that contains the white node(s) (even when there are two white nodes, they lie in one component). The nodes of the diagram obtained define the set I_β of simple roots in Φ. The subgroup G_{I_β} of G is preserved by σ, as is its commutator subgroup G'_{I_β}, and the fixed group $K_\beta = (G'_{I_\beta})^\sigma$ is the desired analogue of G_α. Similarly, $G_{\mathbf{R},\beta}$ is the σ-fixed group in $(G_{\mathbf{C}})_{I_\beta}$. Note that we have selected a sub—Satake diagram corresponding to the rank one symmetric space G_{I_β}/K_β.

Examples. In the split case, identifying Φ with Σ, we have $G_{I_\beta} = G_\beta \cong SU(2)$ and $K_\beta \cong SO(2)$ for all β. For the usual involution on $SU(2n)$ with $K = Sp(n)$, the subdiagrams obtained all have the form $\bullet\!-\!\!-\!\!-\!\circ\!-\!\!-\!\!-\!\bullet$, so $G_{I_\beta} \cong SU(4)$ and $K_\beta \cong Sp(2)$ for all β (§ 6.1).

If β_0 is the highest root of Σ, K_{β_0}, $(G_0)_{\beta_0}$ are similarly defined, using the extended Satake diagram.

Lattices are defined exactly as before, using t_m, T_m and Σ in place of t, T, Φ. The coroot, integral, and coweight lattices for M will be denoted R_m, I_m, J_m, respectively. In fact, in each case the lattice for M is obtained by simply intersecting the corresponding lattice for G (in t) with t_m. The definition of the affine Weyl group $\tilde{W}_{G,K}$, the Stiefel diagram, alcoves, Cartan simplex Δ_m etc. are exactly as above—indeed these depend only on the root system Σ. In fact $\Delta_m = \Delta \cap t_m$. Theorems (1.3) and (1.5) also go through in the following form, for example.

(1.8) THEOREM. *Let G be a simple compact Lie group with involution σ and fixed group K as above. Then every element of M is K-conjugate to an element of the form $\exp X, X \in \Delta_m$. If G/K is simply-connected, X is unique.*

To state the analogue of (1.5), we need to determine $C_K \exp X$ for $X \in (\mathring{\Delta}_m)_I$. Here I is a subset of $\tilde{S}_{\mathbf{R}}$—the set of simple roots of Σ. Clearly $C_K \exp X = (C_G \exp)^\sigma$. It follows easily that $C_K \exp X = (G_{I'})^\sigma$, where I' is obtained from I in the obvious way: In the extended Satake diagram, I' corresponds to the black nodes together with all the white nodes that "restrict" to the nodes of I. (For example, if I is the empty set—i.e., X lies in the interior of the Cartan simplex Δ_m – I' corresponds to the black nodes and $C_K \exp X = (G_{I'})^\sigma = C_K t_m$). Let $K_I = (G_{I'})^\sigma$.

(1.9) THEOREM. *Let $G, \sigma, K,$ be as in (1.8), with $G/K \equiv M$ simply-connected, and regard M as a quotient space of $K/C_K t_m \times \Delta_m$ via the map $(kC_K T_m, X) \mapsto k \exp X\, k^{-1}$. Then the equivalence relation on $K/C_K t_m \times \Delta_m$ is given by $(k_1, X) \sim (k_2, X)$ if $X \in (\mathring{\Delta}_m)_I$ and $k_1 = k_2 \bmod K_I$.*

The final volley in our barrage of notation has to do with Weyl groups. If (W, S) is any Coxeter system, and I is a subset of S, W_I is the subcoxeter system generated by I. Each coset wW_I has a unique element X of minimal length, and $l(xy) = l(x) + l(y)$ for all $y \in W_I$ ($l(w)$ is the length of w as a word in the elements of S). We let W^I denote the set of such minimal length elements. We also recall that W^I has a partial order—the Bruhat order—defined by setting $x \leqslant y$ if y has a reduced decomposition

$y = s_1 \cdots s_k (s_i \in S)$ and x has a reduced decomposition obtained by deleting some subset of the s_i's occuring in y. (For a very nice account of these related matters, see [14]). If W is finite, W has a unique element w_0 of maximal length, we define the length of W to be $l(w_0)$.

§ 2. Topological Buildings

A *Tits system* (G, B, N, S) consists of a group G, subgroups B and N, and a set S, which satisfy the following axioms:

(2.1) $B \cap N$ is normal in N, and S is a set of involutions generating $W \equiv N/B \cap N$,

(2.2) B and N generate G,

(2.3) If $s \in S$, $sBs \neq B$,

(2.4) if $s \in S$, $w \in W$, then $sBw \leqslant BwB \cup BswB$.

(The use of expressions such as sBw is a standard abuse of notation).

Example. Let G be a reductive algebraic group over an algebraically closed field (e.g., $GL(n, \mathbf{C})$), let B be a Borel subgroup (e.g. upper triangular matrices), and let N be the normalizer of a maximal torus (that lies in B). This data determines a set S of simple reflections generating the Weyl group W (e.g., the usual generators $s_1, \ldots, s_{n-1}$ of Σ_n). Then one of the main results in the structure theory of reductive groups is that (G, B, N, S) is a Tits system (see for example [15]).

Throughout this paper we will assume that the set S is finite; its cardinality l is the *rank* of the system.

We next list some of the important properties of a Tits system.

(2.5) (Bruhat Decomposition) $G = \coprod_{w \in W} BwB$ (disjoint union),

(2.6) (W, S) is a Coxeter system.

A subgroup P of G is *parabolic* if it contains a conjugate of B. In particular if $I \subseteq S$, the subgroup P_I generated by B and I is parabolic.

(2.7) (a) The parabolic subgroups containing B are precisely the P_I, $I \subseteq S$. No two of these are conjugate; in particular there are exactly 2^l such subgroups, which form a lattice isomorphic to the lattice of subsets of S.

(b) $P_I = BW_IB$

(c) Every parabolic P is self-normalizing: $N_G P = P$.