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Artikel: ABOUT THE PROOFS OF CALABI'S CONJECTURES ON COMPACT KÄHLER MANIFOLDS

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Kapitel: 9. A PRIORI ESTIMATES OF ORDER FIVE AND MORE

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$$S_{4,4}(z_0) \leqslant c_7 \,,$$

for some controlled constant c_7, and anywhere else on X, since $\theta \leqslant \theta(z_0)$ and $\| D\nabla\bar{\nabla}\varphi \| \leqslant C_3$, one infers that:

$$S_{4,4} \leqslant c_7 \exp(2\varepsilon C_3)\,.$$

9. A PRIORI ESTIMATES OF ORDER FIVE AND MORE

Here, in order to prove 7.1 with $n \geqslant 5$, we consider the functional:

$$S_{n,n} = \frac{1}{2} \sum_{|\alpha| = n-2} \varphi_{a\bar{b}\alpha}\, \varphi_{\bar{a}b\bar{\alpha}}$$

(the coefficient $\frac{1}{2}$ appears for both definitions of $S_{4,4}$ to agree).

Again $S_{n,n}$ is *coercive* and we compute in a similar way,

$$-\Delta'(S_{n,n}) = T_{n+2,n} + T_{n+1,n+1} \qquad (\text{mod. } E_{n-1})\,,$$

where $T_{n+1,n+1}$ is *coercive.* As for $T_{n+2,n}$, proceeding as in the previous section, we find:

$$T_{n+2,n} = T_{n+1,n} + T_{n,n} + T_n \qquad (\text{mod. } E_{n-1})\,.$$

Hence,

$$-\Delta'(S_{n,n}) = T_{n+1,n+1} + T_{n+1,n} + T_{n,n} + T_n \qquad (\text{mod. } E_{n-1})\,,$$

with $T_{n+1,n+1}$ coercive. Changing n into $(n-1)$, for $n \geqslant 6$, yields *still modulo* E_{n-1}

$$-\Delta'(S_{n-1,n-1}) = T'_{n,n} + T'_n \qquad (\text{mod. } E_{n-1})\,.$$

In view of formula (4) of the preceding section, this holds for $n = 5$ as well. From the *coercivity* of $T'_{n,n}$ we may choose constants $c_i > 0$, such that

$$-\Delta'(S_{n-1,n-1}) \geqslant c_1\, S_{n,n} - c_2 (S_{n,n})^{\frac{1}{2}} - c_3\,.$$

Moreover we may choose constants c_i such that

$$| T_{n+1,n} | \leqslant 2c_4 (T_{n+1,n+1}\, S_{n,n})^{\frac{1}{2}}\,, \quad | T_{n,n} | \leqslant c_5\, S_{n,n}\,, \quad | T_n | \leqslant c_6 (S_{n,n})^{\frac{1}{2}}\,,$$

$$\text{and} \qquad c_1 c_7 > c_4^2 + c_5\,.$$

We obtain,

$$-\Delta'(S_{n,n}+c_7 S_{n-1,n-1}) \geqslant (c_1c_7-c_4^2-c_5)S_{n,n} - (c_6+c_2c_7)(S_{n,n})^{\frac{1}{2}} - c_3c_7$$

and the proof may be easily completed.

10. The analytic point of view

Since equation (1) is *elliptic* and g, as a Kähler metric, is real analytic for the underlying real (analytic) structure of X, by the general elliptic regularity theory e.g. [17], p. 266-277 if $P_\lambda(\varphi)$ is real analytic so are φ and g'. Hence a purely analytic proof would be desirable.

Real analytic inverse function theorems are available since the work of J. Nash [19] who made a decisive use of smoothing operators (see also [13]). A theorem of H. Jaccbowitz [15] (p. 203) (see also [25], p. 94-101, 137-138) is available, the proof of which is purely analytical and does not use smoothing operators. This approach was first initiated by A. Kolmogorov (1954) and developed by V. Arnold (1961) (see references in [18]), and by J. Moser [18] (p. 513-533). Unfortunately, the application to nonlinear elliptic operators is not achieved.

A further trouble arises from the fact that the space of analytic functions *is not metrizable.*

Last but not least, we could not carry out analytic *a priori* estimates.

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