

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 34 (1988)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ABOUT THE PROOFS OF CALABI'S CONJECTURES ON COMPACT KÄHLER MANIFOLDS  
**Autor:** Delanoë, Ph. / Hirschowitz, A.  
**Kapitel:** 8. A PRIORI ESTIMATES OF ORDER FOUR  
**DOI:** <https://doi.org/10.5169/seals-56591>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 30.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

$$\begin{aligned}
T_{3,3} + T_2 & \quad \text{when } n = 2, \\
T_{4,3} + T_{3,3,3} + T_3 & \quad \text{when } n = 3, \\
T_5 + T_{4,4} + T_4 & \quad \text{when } n = 4, \\
T_{n+1} + T_n & \quad \text{when } n \geq 5.
\end{aligned}$$

*Proof.* The cases  $n = 2, 3, 4, 5$ , must be checked bare-handed. There is no difficulty. Then, for  $n \geq 5$ , one can proceed by induction on  $n$ . Indeed assume,

$$\Phi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1}, \quad \text{for some } n = |\alpha| \geq 5.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\Phi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \Phi_{ac\alpha} \Phi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n,$$

since  $|ac\alpha| = n + 2$ . The same is true with  $\bar{b}$  instead of  $b$ . Q.E.D.

*Remark 7.7.* The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for  $n = 4$  (in order to kill the effect of the term  $T_{4,4}$ ) and that the same (simpler) procedure should then apply, arguing by iteration, for any  $n \geq 5$ .

Notice also that the hardest case appears to be  $n = 3$ . Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \Phi_{ab'c} \Phi_{a'bc'},$$

perform a careful calculation of  $\Delta'(S_{3,3})$  and use either the Maximum Principle [24] or a recurrence on  $L^p(dX_{g'})$  norms of  $S_{3,3}$  [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case  $n = 3$ .

## 8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with  $n = 4$ , we consider the functional:

$$S_{4,4} = \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} + \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}}.$$

It is enough to estimate  $S_{4,4}$  since it is *coercive*. Let us compute  $-\Delta'(S_{4,4})$ . One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3),$$

where  $T_{5,5}$  is *coercive*, while the sixth order derivatives in  $T_{6,4}$  occur through  $\varphi_{\bar{a}\bar{b}\alpha c c'}$  with  $|\alpha| = 2$ .

In view of 7.4 and 7.6, after bringing the indices  $cc'$  in first position, we get

$$(4) \quad -\Delta'(S_{4,4}) = T_{5,5} + T_{5,4} + T_{4,4,4} + T_{4,4} + T_4 \pmod{E_3}$$

where  $T_{5,5}$  is the *coercive* term from above.

As expected in remark 7.7, in order to control the term  $T_{4,4,4}$ , we need to consider instead of  $S_{4,4}$  another functional, namely:

$$\theta = S_{4,4} \exp(\varepsilon \varphi_{\bar{a}\bar{b}c} \varphi_{\bar{a}\bar{b}\bar{c}}),$$

where  $\varepsilon$  is a constant to be chosen later on. Then we compute the quantity

$$Q = -(\Delta'\theta) \exp(-\varepsilon \varphi_{\bar{a}\bar{b}c} \varphi_{\bar{a}\bar{b}\bar{c}});$$

and we easily find

$$Q = -\Delta'(S_{4,4}) + \varepsilon T_{4,4,4,4} + \varepsilon^2 T'_{4,4,4,4} + \varepsilon T_{5,4,4} \pmod{E_3},$$

where  $T'_{4,4,4,4}$  is a square and where

$$T_{4,4,4,4} = S_{4,4}(\varphi_{\bar{a}\bar{b}c\bar{d}} \varphi_{\bar{a}\bar{b}\bar{c}d'} + \varphi_{\bar{a}\bar{b}c'd} \varphi_{\bar{a}\bar{b}\bar{c}d}).$$

So there exists a constant  $c_1$  such that (see remark 5.1),

$$(S_{4,4})^2 \leq c_1 T_{4,4,4,4}.$$

Furthermore we may choose constants  $c_i$  such that,

$$\begin{aligned} |T_{5,4,4}| &\leq c_2 S_{4,4} (T_{5,5})^{\frac{1}{2}}, & |T_{5,4}| &\leq c_3 (T_{5,5} S_{4,4})^{\frac{1}{2}}, \\ |T_{4,4,4}| &\leq c_4 (S_{4,4})^{\frac{3}{2}}, & |T_{4,4}| &\leq c_5 S_{4,4}, & |T_4| &\leq c_6 (S_{4,4})^{\frac{1}{2}}. \end{aligned}$$

By splitting  $T_{5,5}$  in its two halves and by putting each half together with  $T_{5,4,4}$  and  $T_{5,4}$  respectively, one obtains:

$$Q \geq \left( \frac{\varepsilon}{c_1} - \frac{1}{2} \varepsilon^2 c_2^2 \right) (S_{4,4})^2 - c_4 (S_{4,4})^{\frac{3}{2}} - \left( c_5 + \frac{1}{2} c_3^2 \right) S_{4,4} - c_6 (S_{4,4})^{\frac{1}{2}}.$$

Now  $\varepsilon$  must be chosen small enough in order for the coefficient of  $(S_{4,4})^2$  to be *strictly* positive:  $\varepsilon \in (0, (2/c_1 c_2^2))$ .

To complete the proof, one argues that  $Q(z_0) \leq 0$  at a point  $z_0 \in X$  where  $\theta$  assumes its *maximum* on  $X$ , which implies

$$S_{4,4}(z_0) \leq c_7,$$

for some controlled constant  $c_7$ , and anywhere else on  $X$ , since  $\theta \leq \theta(z_0)$  and  $\|D\bar{\nabla}\bar{\nabla}\varphi\| \leq C_3$ , one infers that:

$$S_{4,4} \leq c_7 \exp(2\varepsilon C_3).$$

## 9. A PRIORI ESTIMATES OF ORDER FIVE AND MORE

Here, in order to prove 7.1 with  $n \geq 5$ , we consider the functional:

$$S_{n,n} = \frac{1}{2} \sum_{|\alpha|=n-2} \varphi_{a\bar{b}\alpha} \varphi_{\bar{a}b\bar{\alpha}}$$

(the coefficient  $\frac{1}{2}$  appears for both definitions of  $S_{4,4}$  to agree).

Again  $S_{n,n}$  is *coercive* and we compute in a similar way,

$$-\Delta'(S_{n,n}) = T_{n+2,n} + T_{n+1,n+1} \pmod{E_{n-1}},$$

where  $T_{n+1,n+1}$  is *coercive*. As for  $T_{n+2,n}$ , proceeding as in the previous section, we find:

$$T_{n+2,n} = T_{n+1,n} + T_{n,n} + T_n \pmod{E_{n-1}}.$$

Hence,

$$-\Delta'(S_{n,n}) = T_{n+1,n+1} + T_{n+1,n} + T_{n,n} + T_n \pmod{E_{n-1}},$$

with  $T_{n+1,n+1}$  *coercive*. Changing  $n$  into  $(n-1)$ , for  $n \geq 6$ , yields *still modulo*  $E_{n-1}$

$$-\Delta'(S_{n-1,n-1}) = T'_{n,n} + T'_n \pmod{E_{n-1}}.$$

In view of formula (4) of the preceding section, this holds for  $n = 5$  as well. From the *coercivity* of  $T'_{n,n}$  we may choose constants  $c_i > 0$ , such that

$$-\Delta'(S_{n-1,n-1}) \geq c_1 S_{n,n} - c_2 (S_{n,n})^{\frac{1}{2}} - c_3.$$

Moreover we may choose constants  $c_i$  such that

$$|T_{n+1,n}| \leq 2c_4 (T_{n+1,n+1} S_{n,n})^{\frac{1}{2}}, \quad |T_{n,n}| \leq c_5 S_{n,n}, \quad |T_n| \leq c_6 (S_{n,n})^{\frac{1}{2}},$$

and  $c_1 c_7 > c_4^2 + c_5$ .