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Artikel: ABOUT THE PROOFS OF CALABI'S CONJECTURES ON COMPACT KÄHLER MANIFOLDS

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Kapitel: 8. A PRIORI ESTIMATES OF ORDER FOUR

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$$\begin{array}{lll} T_{3,3} + T_2 & \textit{when} & n = 2\,, \\ T_{4,3} + T_{3,3,3} + T_3 & \textit{when} & n = 3\,, \\ T_5 + T_{4,4} + T_4 & \textit{when} & n = 4\,, \\ T_{n+1} + T_n & \textit{when} & n \geqslant 5\,. \end{array}$$

Proof. The cases $n = 2, 3, 4, 5$, must be checked bare-handed. There is no difficulty. Then, for $n \geqslant 5$, one can proceed by induction on n. Indeed assume,

$$\varphi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1}\,, \qquad \text{for some } n = |\alpha| \geqslant 5\,.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\varphi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \varphi_{ac\alpha}\,\varphi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n\,,$$

since $|ac\alpha| = n + 2$. The same is true with $\bar{b}$ instead of b. Q.E.D.

Remark 7.7. The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for $n = 4$ (in order to kill the effect of the term $T_{4,4}$) and that the same (simpler) procedure should then apply, arguing by iteration, for any $n \geqslant 5$.

Notice also that the hardest case appears to be $n = 3$. Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \varphi_{ab'c}\,\varphi_{a'bc'}\,,$$

perform a careful calculation of $\Delta'(S_{3,3})$ and use either the Maximum Principle [24] or a recurrence on $L^p(dX_{g'})$ norms of $S_{3,3}$ [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case $n = 3$.

8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with $n = 4$, we consider the functional:

$$S_{4,4} = \varphi_{a\bar{b}cd}\,\varphi_{\bar{a}b\bar{c}\bar{d}} + \varphi_{a\bar{b}\bar{c}d}\,\varphi_{\bar{a}bc\bar{d}}\,.$$

It is enough to estimate $S_{4,4}$ since it is *coercive.* Let us compute $-\Delta'(S_{4,4})$. One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3)\,,$$

where $T_{5,5}$ is *coercive*, while the sixth order derivatives in $T_{6,4}$ occur through $\varphi_{a\bar{b}\alpha c c'}$ with $|\alpha| = 2$.

In view of 7.4 and 7.6, after bringing the indices cc' in first position, we get

$$(4) \qquad -\Delta'(S_{4,4}) = T_{5,5} + T_{5,4} + T_{4,4,4} + T_{4,4} + T_4 \qquad (\text{mod. } E_3)$$

where $T_{5,5}$ is the *coercive* term from above.

As expected in remark 7.7, in order to control the term $T_{4,4,4}$, we need to consider instead of $S_{4,4}$ another functional, namely:

$$\theta = S_{4,4} \exp(\varepsilon\, \varphi_{a\bar{b}c}\, \varphi_{\bar{a}b\bar{c}}),$$

where ε is a constant to be chosen later on. Then we compute the quantity

$$Q = -(\Delta'\theta) \exp(-\varepsilon \varphi_{a\bar{b}c}\, \varphi_{\bar{a}b\bar{c}});$$

and we easily find

$$Q = -\Delta'(S_{4,4}) + \varepsilon T_{4,4,4,4} + \varepsilon^2 T'_{4,4,4,4} + \varepsilon T_{5,4,4} \qquad (\text{mod. } E_3),$$

where $T'_{4,4,4,4}$ is a square and where

$$T_{4,4,4,4} = S_{4,4}(\varphi_{a\bar{b}cd}\, \varphi_{\bar{a}b\bar{c}d'} + \varphi_{a\bar{b}cd'}\, \varphi_{\bar{a}b\bar{c}d}).$$

So there exists a constant c_1 such that (see remark 5.1),

$$(S_{4,4})^2 \leqslant c_1\, T_{4,4,4,4}.$$

Furthermore we may choose constants c_i such that,

$$|T_{5,4,4}| \leqslant c_2\, S_{4,4}(T_{5,5})^{\frac{1}{2}}, \quad |T_{5,4}| \leqslant c_3(T_{5,5}\, S_{4,4})^{\frac{1}{2}},$$

$$|T_{4,4,4}| \leqslant c_4(S_{4,4})^{\frac{3}{2}}, \quad |T_{4,4}| \leqslant c_5 S_{4,4}, \quad |T_4| \leqslant c_6(S_{4,4})^{\frac{1}{2}}.$$

By splitting $T_{5,5}$ in its two halves and by putting each half together with $T_{5,4,4}$ and $T_{5,4}$ respectively, one obtains:

$$Q \geqslant \left(\frac{\varepsilon}{c_1} - \frac{1}{2}\varepsilon^2 c_2^2\right)(S_{4,4})^2 - c_4(S_{4,4})^{\frac{3}{2}} - \left(c_5 + \frac{1}{2}c_3^2\right) S_{4,4} - c_6(S_{4,4})^{\frac{1}{2}}.$$

Now ε must be chosen small enough in order for the coefficient of $(S_{4,4})^2$ to be *strictly* positive: $\varepsilon \in (0, (2/c_1\, c_2^2))$.

To complete the proof, one argues that $Q(z_0) \leqslant 0$ at a point $z_0 \in X$ where θ assumes its *maximum* on X, which implies

$$S_{4,4}(z_0) \leqslant c_7 \, ,$$

for some controlled constant c_7, and anywhere else on X, since $\theta \leqslant \theta(z_0)$ and $\| D\nabla\bar{\nabla}\varphi \| \leqslant C_3$, one infers that:

$$S_{4,4} \leqslant c_7 \exp(2\varepsilon C_3) \, .$$

9. A priori estimates of order five and more

Here, in order to prove 7.1 with $n \geqslant 5$, we consider the functional:

$$S_{n,n} = \frac{1}{2} \sum_{|\alpha| = n-2} \varphi_{a\bar{b}\alpha} \, \varphi_{\bar{a}b\bar{\alpha}}$$

(the coefficient $\frac{1}{2}$ appears for both definitions of $S_{4,4}$ to agree).

Again $S_{n,n}$ is *coercive* and we compute in a similar way,

$$-\Delta'(S_{n,n}) = T_{n+2,n} + T_{n+1,n+1} \quad (\text{mod. } E_{n-1}) \, ,$$

where $T_{n+1,n+1}$ is *coercive.* As for $T_{n+2,n}$, proceeding as in the previous section, we find:

$$T_{n+2,n} = T_{n+1,n} + T_{n,n} + T_n \quad (\text{mod. } E_{n-1}) \, .$$

Hence,

$$-\Delta'(S_{n,n}) = T_{n+1,n+1} + T_{n+1,n} + T_{n,n} + T_n \quad (\text{mod. } E_{n-1}) \, ,$$

with $T_{n+1,n+1}$ coercive. Changing n into $(n-1)$, for $n \geqslant 6$, yields *still modulo* E_{n-1}

$$-\Delta'(S_{n-1,n-1}) = T'_{n,n} + T'_n \quad (\text{mod. } E_{n-1}) \, .$$

In view of formula (4) of the preceding section, this holds for $n = 5$ as well. From the *coercivity* of $T'_{n,n}$ we may choose constants $c_i > 0$, such that

$$-\Delta'(S_{n-1,n-1}) \geqslant c_1 \, S_{n,n} - c_2 (S_{n,n})^{\frac{1}{2}} - c_3 \, .$$

Moreover we may choose constants c_i such that

$$| T_{n+1,n} | \leqslant 2c_4 (T_{n+1,n+1} \, S_{n,n})^{\frac{1}{2}} \, , \quad | T_{n,n} | \leqslant c_5 \, S_{n,n} \, , \quad | T_n | \leqslant c_6 (S_{n,n})^{\frac{1}{2}} \, ,$$

$$\text{and} \qquad c_1 c_7 > c_4^2 + c_5 \, .$$