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We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to g or to g' according to whether the letter occurs with a bar or with a prime. So,

$$T_{\dots a \dots \bar{a} \dots} \text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \text{ while}$$

$$T_{\dots a \dots a' \dots} \text{ stands for } g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots} .$$

As usual if $T_{a\dots l}$ is a tensor, further lower indices refer to covariant differentiation (with respect to g); so,

$$T_{a\dots lm} \text{ stands for } \nabla_m T_{a\dots l}, \text{ while}$$

$$T_{a\dots l\bar{m}} \text{ stands for } \bar{\nabla}_{\bar{m}} T_{a\dots l} .$$

Our indices will be latin letters; greek letters will denote multi-indices. If α is a multi-index, $\bar{\alpha}$ will denote the *conjugate* multi-index (for instance if $\alpha = \bar{a}\bar{b}\bar{c}$, then $\bar{\alpha} = \bar{a}\bar{b}\bar{c}$), while $|\alpha|$ denotes its length. We shall say that α is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations $D, \nabla, \bar{\nabla}, \parallel, \parallel$, were introduced in section 4.

Remark 6.1. Since covariant differentiation (with respect to g) and contraction with respect to g' *do not* commute, we observe that, for instance, the difference (recall $g' = g + \nabla\bar{\nabla}\phi$)

$$(3) \quad \phi_{aa'ab} - (\phi_{aa'\alpha})_b \equiv \phi_{ac\alpha} \phi_{a'c'b}$$

does not vanish.

7. HIGHER ORDER A PRIORI ESTIMATES: GENERALITIES

We want to prove by induction,

PROPOSITION 7.1. *Given $n \geq 4$, a sequence $(K_i), i \in \mathbf{N}$, and a finite sequence C_0, \dots, C_{n-1} , there exists C_n such that:*

$$\|\phi\| \leq C_0, \quad \forall i = 0, \dots, n-3, \quad \|D^i \nabla \bar{\nabla} \phi\| \leq C_{i+2}$$

and $\forall i \in \mathbf{N}, \quad \|D^i P_\lambda(\phi)\| \leq K_i,$

implies

$$\|D^{n-2} \nabla \bar{\nabla} \phi\| \leq C_n .$$

Actually one needs $\|D^i P_\lambda(\varphi)\| \leq K_i$ only for $0 \leq i \leq n$, hence C_n depends only upon $(C_0, \dots, C_{n-1}, K_0, \dots, K_n)$.

Hereafter, by "a constant", we will mean a constant which depends only upon the given constants $(C_0, \dots, C_{n-1}, K_0, \dots, K_n)$.

Let us explain a further convention.

Convention 7.2. We will have to consider sums of tensors obtained via contractions of tensor polynomials in the variables $(g')^{-1}, \nabla\bar{\nabla}\varphi, \dots, D^i\nabla\bar{\nabla}\varphi, \dots$. The present convention helps describing the variables occurring in (still) uncontrolled expressions.

First of all, given $\varphi \in A_\lambda$ and an integer $n \geq 3$, we denote by E_{n-1} the (finite dimensional complex) vector space generated by all contracted tensor polynomials, with degree of homogeneity at most $2n$, in the variables

$$(g')^{-1}, \nabla\bar{\nabla}\varphi, D\nabla\bar{\nabla}\varphi, \dots, D^{n-3}\nabla\bar{\nabla}\varphi, D^i P_\lambda(\varphi), \quad i = 0, \dots, n.$$

In order to prove 7.1, we will compute *modulo* E_{n-1} .

Given integers p, \dots, s , all of them $\geq n$, we will say that mod. E_{n-1} a tensor T is "of the form $T_{p, \dots, s}$ ", whenever mod. E_{n-1} it is a sum of contractions of tensors

$$A \otimes D^{p-2}\nabla\bar{\nabla}\varphi \otimes \dots \otimes D^{s-2}\nabla\bar{\nabla}\varphi,$$

where the A 's are in E_{n-1} .

Furthermore for $s \geq n$, under the assumptions of 7.1, we will say that a scalar term $T_{s,s}$ is *coercive*, if for any other term of the form T'_s (resp. $T''_{s,s}$) there exists a constant C such that:

$$|T'_s| \leq C(T_{s,s})^{\frac{1}{2}} \quad (\text{resp. } |T''_{s,s}| \leq CT_{s,s}).$$

We present now three lemmas which illustrate the previous convention.

LEMMA 7.3. *Given integers $s \geq n \geq 3$, the covariant derivative (in metric g) of a term of the form T_s mod. E_{n-1} , is of the form $(T_{s+1} + T_s)$ mod. E_n .*

Proof. This is just because the derivative $D[(g')^{-1}]$ is a contracted tensor polynomial (of degree 3) in $(g')^{-1}$ and $D\nabla\bar{\nabla}\varphi$.

LEMMA 7.4. *If α and β are two distinct mixed multi-indices of length $(n+2)$ obtained from each other by permutation, then the difference of covariant derivatives $(\varphi_\alpha - \varphi_\beta)$ is of the form T_n mod. E_{n-1} .*

Proof. On the Kähler manifold (X, g) , commuting two consecutive covariant derivatives yields curvature terms only if the couple of derivatives concerned is *mixed* (for general commutation rules on Riemannian manifolds see e.g. [21], exposé XI, proposition 3.2). If so, say k and \bar{l} are the permuted indices, the result will involve

$$R_{p\bar{k}\bar{l}}^q \quad (\text{curvature tensor of } g)$$

with p and q of the same type. Explicitely:

$$\Phi_{\lambda k \bar{l} \mu} - \Phi_{\lambda \bar{l} k \mu} = \sum_p R_{p\bar{q}k\bar{l}} \Phi_{\nu q \tau}$$

for all p, ν, τ , such that $\nu p \tau \equiv \lambda \mu$. Hence the *types* of all the remaining non-permuted covariant derivatives $\Phi_{\nu q \tau}$ are *identically preserved*. In particular if γ and δ denote two multi-indices of length n obtained from each other by permutation, necessarily

$$(\Phi_{i\bar{j}\gamma} - \Phi_{i\bar{j}\delta}) \text{ is of the form } T_n \text{ mod. } E_{n-1},$$

since two *mixed* derivatives will keep bearing in first place on φ in the process of permutation.

The proof of lemma 7.4 is therefore reduced to the following two cases for the multi-indices α and β :

$$\begin{aligned} \text{either } & \alpha = i\bar{j}k\lambda, \quad \beta = k\bar{j}i\lambda, \quad |\lambda| = n - 1, \\ \text{or } & \alpha = i\bar{j}k\bar{l}\mu, \quad \beta = k\bar{l}i\bar{j}\mu, \quad |\mu| = n - 2. \end{aligned}$$

In the first case, one has identically on a Kähler manifold:

$$\Phi_\alpha - \Phi_\beta \equiv 0.$$

In the second case, the same reasoning as above holds for $(\Phi_\alpha - \Phi_\beta)$ since it can be written as

$$(\Phi_{i\bar{j}k\bar{l}\mu} - \Phi_{ik\bar{j}\bar{l}\mu}) + (\Phi_{ki\bar{l}\bar{j}\mu} - \Phi_{k\bar{l}i\bar{j}\mu}),$$

each of these two commutations being clearly of the form $T_n \text{ mod. } E_{n-1}$.
Q.E.D.

Remark 7.5. The fact that commutation formulae involve only *mixed* derivatives was already a crucial detail in the proofs of the second and third order *a priori* estimates.

LEMMA 7.6. *The tensor $\Phi_{aa'\alpha}$ where α is a mixed multi-index of length n is, mod. E_{n-1} , of the form:*

$$\begin{aligned}
T_{3,3} + T_2 & \quad \text{when } n = 2, \\
T_{4,3} + T_{3,3,3} + T_3 & \quad \text{when } n = 3, \\
T_5 + T_{4,4} + T_4 & \quad \text{when } n = 4, \\
T_{n+1} + T_n & \quad \text{when } n \geq 5.
\end{aligned}$$

Proof. The cases $n = 2, 3, 4, 5$, must be checked bare-handed. There is no difficulty. Then, for $n \geq 5$, one can proceed by induction on n . Indeed assume,

$$\Phi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1}, \quad \text{for some } n = |\alpha| \geq 5.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\Phi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \Phi_{ac\alpha} \Phi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n,$$

since $|ac\alpha| = n + 2$. The same is true with \bar{b} instead of b . Q.E.D.

Remark 7.7. The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for $n = 4$ (in order to kill the effect of the term $T_{4,4}$) and that the same (simpler) procedure should then apply, arguing by iteration, for any $n \geq 5$.

Notice also that the hardest case appears to be $n = 3$. Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \Phi_{ab'c} \Phi_{a'bc'},$$

perform a careful calculation of $\Delta'(S_{3,3})$ and use either the Maximum Principle [24] or a recurrence on $L^p(dX_{g'})$ norms of $S_{3,3}$ [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case $n = 3$.

8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with $n = 4$, we consider the functional:

$$S_{4,4} = \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} + \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} \Phi_{\bar{a}\bar{b}\bar{c}\bar{d}}.$$

It is enough to estimate $S_{4,4}$ since it is *coercive*. Let us compute $-\Delta'(S_{4,4})$. One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3),$$