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We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to  $g$  or to  $g'$  according to whether the letter occurs with a bar or with a prime. So,

$T_{\dots a \dots \bar{a} \dots}$  stands for  $g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}$ , while

$T_{\dots a \dots a' \dots}$  stands for  $g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}$ .

As usual if  $T_{a \dots l}$  is a tensor, further lower indices refer to covariant differentiation (with respect to  $g$ ); so,

$T_{a \dots l m}$  stands for  $\nabla_m T_{a \dots l}$ , while

$T_{a \dots l \bar{m}}$  stands for  $\bar{\nabla}_{\bar{m}} T_{a \dots l}$ .

Our indices will be latin letters; greek letters will denote multi-indices. If  $\alpha$  is a multi-index,  $\bar{\alpha}$  will denote the *conjugate* multi-index (for instance if  $\alpha = a\bar{b}c$ , then  $\bar{\alpha} = \bar{a}b\bar{c}$ ), while  $|\alpha|$  denotes its length. We shall say that  $\alpha$  is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations  $D, \nabla, \bar{\nabla}, \parallel, \parallel$ , were introduced in section 4.

*Remark 6.1.* Since covariant differentiation (with respect to  $g$ ) and contraction with respect to  $g'$  *do not* commute, we observe that, for instance, the difference (recall  $g' = g + \nabla \bar{\nabla} \phi$ )

$$(3) \quad \phi_{aa'ab} - (\phi_{aa'\alpha})_b \equiv \phi_{ac\alpha} \phi_{a'c'b}$$

does not vanish.

## 7. HIGHER ORDER A PRIORI ESTIMATES: GENERALITIES

We want to prove by induction,

PROPOSITION 7.1. *Given  $n \geq 4$ , a sequence  $(K_i), i \in \mathbb{N}$ , and a finite sequence  $C_0, \dots, C_{n-1}$ , there exists  $C_n$  such that:*

$$\|\phi\| \leq C_0, \quad \forall i = 0, \dots, n-3, \quad \|D^i \nabla \bar{\nabla} \phi\| \leq C_{i+2}$$

$$\text{and } \forall i \in \mathbb{N}, \quad \|D^i P_\lambda(\phi)\| \leq K_i,$$

*implies*

$$\|D^{n-2} \nabla \bar{\nabla} \phi\| \leq C_n.$$

Actually one needs  $\|D^i P_\lambda(\varphi)\| \leq K_i$  only for  $0 \leq i \leq n$ , hence  $C_n$  depends only upon  $(C_0, \dots, C_{n-1}, K_0, \dots, K_n)$ .

Hereafter, by "a constant", we will mean a constant which depends only upon the given constants  $(C_0, \dots, C_{n-1}, K_0, \dots, K_n)$ .

Let us explain a further convention.

*Convention 7.2.* We will have to consider sums of tensors obtained via contractions of tensor polynomials in the variables  $(g')^{-1}, \nabla \bar{\nabla} \varphi, \dots, D^i \nabla \bar{\nabla} \varphi, \dots$ . The present convention helps describing the variables occuring in (still) uncontrolled expressions.

First of all, given  $\varphi \in A_\lambda$  and an integer  $n \geq 3$ , we denote by  $E_{n-1}$  the (finite dimensional complex) vector space generated by all contracted tensor polynomials, with degree of homogeneity at most  $2n$ , in the variables

$$(g')^{-1}, \nabla \bar{\nabla} \varphi, D \nabla \bar{\nabla} \varphi, \dots, D^{n-3} \nabla \bar{\nabla} \varphi, D^i P_\lambda(\varphi), \quad i = 0, \dots, n.$$

In order to prove 7.1, we will compute *modulo*  $E_{n-1}$ .

Given integers  $p, \dots, s$ , all of them  $\geq n$ , we will say that *mod.*  $E_{n-1}$  a tensor  $T$  is "of the form  $T_{p, \dots, s}$ ", whenever *mod.*  $E_{n-1}$  it is a sum of contractions of tensors

$$A \otimes D^{p-2} \nabla \bar{\nabla} \varphi \otimes \dots \otimes D^{s-2} \nabla \bar{\nabla} \varphi,$$

where the  $A$ 's are in  $E_{n-1}$ .

Furthermore for  $s \geq n$ , under the assumptions of 7.1, we will say that a scalar term  $T_{s,s}$  is *coercive*, if for any other term of the form  $T'_s$  (resp.  $T''_{s,s}$ ) there exists a constant  $C$  such that:

$$|T'_s| \leq C(T_{s,s})^{\frac{1}{2}} \quad (\text{resp. } |T''_{s,s}| \leq C T_{s,s}).$$

We present now three lemmas which illustrate the previous convention.

**LEMMA 7.3.** *Given integers  $s \geq n \geq 3$ , the covariant derivative (in metric  $g$ ) of a term of the form  $T_s \bmod. E_{n-1}$ , is of the form  $(T_{s+1} + T_s) \bmod. E_n$ .*

*Proof.* This is just because the derivative  $D[(g')^{-1}]$  is a contracted tensor polynomial (of degree 3) in  $(g')^{-1}$  and  $D \nabla \bar{\nabla} \varphi$ .

**LEMMA 7.4.** *If  $\alpha$  and  $\beta$  are two distinct mixed multi-indices of length  $(n+2)$  obtained from each other by permutation, then the difference of covariant derivatives  $(\varphi_\alpha - \varphi_\beta)$  is of the form  $T_n \bmod. E_{n-1}$ .*

*Proof.* On the Kähler manifold  $(X, g)$ , commuting two consecutive covariant derivatives yields curvature terms only if the couple of derivatives concerned is *mixed* (for general commutation rules on Riemannian manifolds see e.g. [21], exposé XI, proposition 3.2). If so, say  $k$  and  $\bar{l}$  are the permuted indices, the result will involve

$$R_{p\bar{k}l}^q \quad (\text{curvature tensor of } g)$$

with  $p$  and  $q$  of the same type. Explicitely:

$$\varphi_{\lambda k \bar{l} \mu} - \varphi_{\lambda \bar{l} k \mu} = \sum_p R_{p\bar{q}k\bar{l}} \varphi_{\nu q \tau}$$

for all  $p, \nu, \tau$ , such that  $\nu p \tau \equiv \lambda \mu$ . Hence the types of all the remaining non-permuted covariant derivatives  $\varphi_{\nu q \tau}$  are *identically preserved*. In particular if  $\gamma$  and  $\delta$  denote two multi-indices of length  $n$  obtained from each other by permutation, necessarily

$$(\varphi_{i\bar{j}\gamma} - \varphi_{i\bar{j}\delta}) \text{ is of the form } T_n \text{ mod. } E_{n-1},$$

since two *mixed* derivatives will keep bearing in first place on  $\varphi$  in the process of permutation.

The proof of lemma 7.4 is therefore reduced to the following two cases for the multi-indices  $\alpha$  and  $\beta$ :

$$\begin{aligned} \text{either } \alpha &= i\bar{j}k\lambda, \quad \beta = k\bar{j}i\lambda, \quad |\lambda| = n-1, \\ \text{or } \alpha &= i\bar{j}k\bar{l}\mu, \quad \beta = k\bar{l}i\bar{j}\mu, \quad |\mu| = n-2. \end{aligned}$$

In the first case, one has identically on a Kähler manifold:

$$\varphi_\alpha - \varphi_\beta \equiv 0.$$

In the second case, the same reasoning as above holds for  $(\varphi_\alpha - \varphi_\beta)$  since it can be written as

$$(\varphi_{i\bar{j}k\bar{l}\mu} - \varphi_{ik\bar{j}l\mu}) + (\varphi_{ki\bar{l}j\mu} - \varphi_{k\bar{l}i\bar{j}\mu}),$$

each of these two commutations being clearly of the form  $T_n \text{ mod. } E_{n-1}$ .  
Q.E.D.

*Remark 7.5.* The fact that commutation formulae involve only *mixed* derivatives was already a crucial detail in the proofs of the second and third order *a priori* estimates.

LEMMA 7.6. The tensor  $\varphi_{aa'\alpha}$  where  $\alpha$  is a mixed multi-index of length  $n$  is, mod.  $E_{n-1}$ , of the form:

$$\begin{array}{ll}
T_{3,3} + T_2 & \text{when } n = 2, \\
T_{4,3} + T_{3,3,3} + T_3 & \text{when } n = 3, \\
T_5 + T_{4,4} + T_4 & \text{when } n = 4, \\
T_{n+1} + T_n & \text{when } n \geq 5.
\end{array}$$

*Proof.* The cases  $n = 2, 3, 4, 5$ , must be checked bare-handed. There is no difficulty. Then, for  $n \geq 5$ , one can proceed by induction on  $n$ . Indeed assume,

$$\varphi_{aa'\alpha} = T_{n+1} + T_n \bmod. E_{n-1}, \quad \text{for some } n = |\alpha| \geq 5.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\varphi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \varphi_{ac\alpha} \varphi_{a'c'b} = T_{n+2} + T_{n+1} \bmod. E_n,$$

since  $|ac\alpha| = n + 2$ . The same is true with  $\bar{b}$  instead of  $b$ . Q.E.D.

*Remark 7.7.* The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for  $n = 4$  (in order to kill the effect of the term  $T_{4,4}$ ) and that the same (simpler) procedure should then apply, arguing by iteration, for any  $n \geq 5$ .

Notice also that the hardest case appears to be  $n = 3$ . Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \varphi_{ab'c} \varphi_{a'bc'},$$

perform a careful calculation of  $\Delta'(S_{3,3})$  and use either the Maximum Principle [24] or a recurrence on  $L^p(dX_{g'})$  norms of  $S_{3,3}$  [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case  $n = 3$ .

## 8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with  $n = 4$ , we consider the functional:

$$S_{4,4} = \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}} \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}} + \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}} \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}}.$$

It is enough to estimate  $S_{4,4}$  since it is *coercive*. Let us compute  $-\Delta'(S_{4,4})$ . One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\bmod. E_3),$$