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We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to g or to g' according to whether the letter occurs with a bar or with a prime. So,

$$T_{\dots a \dots \bar{a} \dots} \text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \quad \text{while}$$
$$T_{\dots a \dots a' \dots} \text{ stands for } g'^{\,a\bar{b}} T_{\dots a \dots \bar{b} \dots} \quad .$$

As usual if $T_{a \dots l}$ is a tensor, further lower indices refer to covariant differentiation (with respect to g); so,

$$T_{a \dots lm} \text{ stands for } \nabla_m T_{a \dots l}, \quad \text{while}$$
$$T_{a \dots l\bar{m}} \text{ stands for } \bar{\nabla}_{\bar{m}} T_{a \dots l} \quad .$$

Our indices will be latin letters; greek letters will denote multi-indices. If α is a multi-index, $\bar{\alpha}$ will denote the *conjugate* multi-index (for instance if $\alpha = a\bar{b}c$, then $\bar{\alpha} = \bar{a}b\bar{c}$), while $|\alpha|$ denotes its length. We shall say that α is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations D, ∇, $\bar{\nabla}$, $\| \ \|$, were introduced in section 4.

Remark 6.1. Since covariant differentiation (with respect to g) and contraction with respect to g' *do not* commute, we observe that, for instance, the difference (recall $g' = g + \nabla\bar{\nabla}\varphi$)

$$(3) \qquad \varphi_{aa'\alpha b} - (\varphi_{aa'\alpha})_b \equiv \varphi_{ac\alpha}\,\varphi_{a'c'b}$$

does not vanish.

7. Higher order a priori estimates: generalities

We want to prove by induction,

Proposition 7.1. *Given* $n \geqslant 4$, *a sequence* (K_i), $i \in \mathbf{N}$, *and a finite sequence* $C_0, \dots, C_{n-1}$, *there exists* C_n *such that:*

$$\| \varphi \| \leqslant C_0, \quad \forall i = 0, \dots, n-3, \quad \| D^i \nabla\bar{\nabla}\varphi \| \leqslant C_{i+2}$$
$$\textit{and} \quad \forall i \in \mathbf{N}, \quad \| D^i P_\lambda(\varphi) \| \leqslant K_i,$$

implies

$$\| D^{n-2} \nabla\bar{\nabla}\varphi \| \leqslant C_n .$$

Actually one needs $\| D^i P_\lambda(\varphi) \| \leqslant K_i$ *only for* $0 \leqslant i \leqslant n$, *hence* C_n *depends only upon* $(C_0, ..., C_{n-1}, K_0, ..., K_n)$.

Hereafter, by "a constant", we will mean a constant which depends only upon the given constants $(C_0, ..., C_{n-1}, K_0, ..., K_n)$.

Let us explain a further convention.

Convention 7.2. We will have to consider sums of tensors obtained via contractions of tensor polynomials in the variables $(g')^{-1}$, $\nabla\bar{\nabla}\varphi, ..., D^i\nabla\bar{\nabla}\varphi, ...$. The present convention helps describing the variables occuring in (still) uncontrolled expressions.

First of all, *given* $\varphi \in A_\lambda$ and an integer $n \geqslant 3$, we denote by E_{n-1} the (finite dimensional complex) vector space generated by all contracted tensor polynomials, with degree of homogeneity at most $2n$, in the variables

$$(g')^{-1}, \quad \nabla\bar{\nabla}\varphi, \quad D\nabla\bar{\nabla}\varphi, ..., D^{n-3}\nabla\bar{\nabla}\varphi, \quad D^i P_\lambda(\varphi), \quad i = 0, ..., n.$$

In order to prove 7.1, we will compute *modulo* E_{n-1}.

Given integers $p, ..., s$, all of them $\geqslant n$, we will say that mod. E_{n-1} a tensor T is "of the form $T_{p,...,s}$", whenever mod. E_{n-1} it is a sum of contractions of tensors

$$A \otimes D^{p-2}\nabla\bar{\nabla}\varphi \otimes ... \otimes D^{s-2}\nabla\bar{\nabla}\varphi,$$

where the A's are in E_{n-1}.

Furthermore for $s \geqslant n$, under the assumptions of 7.1, we will say that a *scalar* term $T_{s,s}$ is *coercive*, if for any other term of the form T'_s (resp. $T''_{s,s}$) there exists a constant C such that:

$$| T'_s | \leqslant C(T_{s,s})^{\frac{1}{2}} \quad (\text{resp. } | T''_{s,s} | \leqslant C T_{s,s}).$$

We present now three lemmas which illustrate the previous convention.

LEMMA 7.3. *Given integers* $s \geqslant n \geqslant 3$, *the covariant derivative (in metric g) of a term of the form* T_s mod. E_{n-1}, *is of the form* $(T_{s+1} + T_s)$ mod. E_n.

Proof. This is just because the derivative $D[(g')^{-1}]$ is a contracted tensor polynomial (of degree 3) in $(g')^{-1}$ and $D\nabla\bar{\nabla}\varphi$.

LEMMA 7.4. *If* $\dot{\alpha}$ *and* β *are two distinct mixed multi-indices of length* $(n+2)$ *obtained from each other by permutation, then the difference of covariant derivatives* $(\varphi_\alpha - \varphi_\beta)$ *is of the form* T_n mod. E_{n-1}.

Proof. On the Kähler manifold (X, g), commuting two consecutive covariant derivatives yields curvature terms only if the couple of derivatives concerned is *mixed* (for general commutation rules on Riemannian manifolds see e.g. [21], exposé XI, proposition 3.2). If so, say k and $\bar{l}$ are the permuted indices, the result will involve

$$R^{q}_{pk\bar{l}} \quad \text{(curvature tensor of } g)$$

with p and q *of the same type*. Explicitely:

$$\varphi_{\lambda k\bar{l}\mu} - \varphi_{\lambda\bar{l}k\mu} = \sum_{p} R_{p\bar{q}k\bar{l}}\, \varphi_{\nu q\tau}$$

for all p, ν, τ, such that $\nu p\tau \equiv \lambda\mu$. Hence the *types* of all the remaining non-permuted covariant derivatives $\varphi_{\nu q\tau}$ are *identically preserved.* In particular if γ and δ denote two multi-indices of length n obtained from each other by permutation, necessarily

$$(\varphi_{i\bar{j}\gamma} - \varphi_{i\bar{j}\delta}) \text{ is of the form } T_n \text{ mod. } E_{n-1}\,,$$

since two *mixed* derivatives will keep bearing in first place on φ in the process of permutation.

The proof of lemma 7.4 is therefore reduced to the following two cases for the multi-indices α and β:

$$\begin{array}{llll} \text{either} & \alpha = i\bar{j}k\lambda\,, & \beta = k\bar{j}i\lambda\,, & |\lambda| = n-1\,, \\ \text{or} & \alpha = i\bar{j}k\bar{l}\mu\,, & \beta = k\bar{l}i\bar{j}\mu\,, & |\mu| = n-2\,. \end{array}$$

In the first case, one has identically on a Kähler manifold:

$$\varphi_{\alpha} - \varphi_{\beta} \equiv 0\,.$$

In the second case, the same reasoning as above holds for $(\varphi_{\alpha} - \varphi_{\beta})$ since it can be written as

$$(\varphi_{i\bar{j}k\bar{l}\mu} - \varphi_{ik\bar{j}\bar{l}\mu}) + (\varphi_{ki\bar{l}\bar{j}\mu} - \varphi_{k\bar{l}i\bar{j}\mu})\,,$$

each of these two commutations being clearly of the form T_n mod. E_{n-1}.

Q.E.D.

Remark 7.5. The fact that commutation formulae involve only *mixed* derivatives was already a crucial detail in the proofs of the second and third order *a priori* estimates.

LEMMA 7.6. *The tensor* $\varphi_{aa'\alpha}$ *where* α *is a mixed multi-index of length* n *is,* mod. E_{n-1}, *of the form:*

$$\begin{array}{lll} T_{3,3} + T_2 & \text{when} & n = 2\,, \\ T_{4,3} + T_{3,3,3} + T_3 & \text{when} & n = 3\,, \\ T_5 + T_{4,4} + T_4 & \text{when} & n = 4\,, \\ T_{n+1} + T_n & \text{when} & n \geqslant 5\,. \end{array}$$

Proof. The cases $n = 2, 3, 4, 5$, must be checked bare-handed. There is no difficulty. Then, for $n \geqslant 5$, one can proceed by induction on n. Indeed assume,

$$\varphi_{aa'\alpha} = T_{n+1} + T_n \text{ mod. } E_{n-1}\,, \qquad \text{for some } n = |\alpha| \geqslant 5\,.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\varphi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \varphi_{ac\alpha}\,\varphi_{a'c'b} = T_{n+2} + T_{n+1} \text{ mod. } E_n\,,$$

since $|ac\alpha| = n + 2$. The same is true with $\bar{b}$ instead of b. Q.E.D.

Remark 7.7. The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for $n = 4$ (in order to kill the effect of the term $T_{4,4}$) and that the same (simpler) procedure should then apply, arguing by iteration, for any $n \geqslant 5$.

Notice also that the hardest case appears to be $n = 3$. Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \varphi_{ab'c}\,\varphi_{a'bc'}\,,$$

perform a careful calculation of $\Delta'(S_{3,3})$ and use either the Maximum Principle [24] or a recurrence on $L^p(dX_{g'})$ norms of $S_{3,3}$ [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case $n = 3$.

8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with $n = 4$, we consider the functional:

$$S_{4,4} = \varphi_{a\bar{b}cd}\,\varphi_{\bar{a}b\bar{c}\bar{d}} + \varphi_{a\bar{b}\bar{c}d}\,\varphi_{\bar{a}bc\bar{d}}\,.$$

It is enough to estimate $S_{4,4}$ since it is *coercive.* Let us compute $-\Delta'(S_{4,4})$. One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\text{mod. } E_3)\,,$$