

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 34 (1988)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ABOUT THE PROOFS OF CALABI'S CONJECTURES ON COMPACT KÄHLER MANIFOLDS
Autor: Delanoë, Ph. / Hirschowitz, A.
Kapitel: 7. HIGHER ORDER A PRIORI ESTIMATES: GENERALITIES
DOI: <https://doi.org/10.5169/seals-56591>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 25.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

We extend the summation convention as follows: we will be concerned only with lower indices. If a letter occurs twice, it refers to a contraction, which is taken with respect to g or to g' according to whether the letter occurs with a bar or with a prime. So,

$$\begin{aligned} T_{\dots a \dots \bar{a} \dots} &\text{ stands for } g^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}, \quad \text{while} \\ T_{\dots a \dots a' \dots} &\text{ stands for } g'^{a\bar{b}} T_{\dots a \dots \bar{b} \dots}. \end{aligned}$$

As usual if $T_{a\dots l}$ is a tensor, further lower indices refer to covariant differentiation (with respect to g); so,

$$\begin{aligned} T_{a\dots lm} &\text{ stands for } \nabla_m T_{a\dots l}, \quad \text{while} \\ T_{a\dots l\bar{m}} &\text{ stands for } \bar{\nabla}_{\bar{m}} T_{a\dots l}. \end{aligned}$$

Our indices will be latin letters; greek letters will denote multi-indices. If α is a multi-index, $\bar{\alpha}$ will denote the *conjugate* multi-index (for instance if $\alpha = a\bar{b}c$, then $\bar{\alpha} = \bar{a}b\bar{c}$), while $|\alpha|$ denotes its length. We shall say that α is *mixed* if its length is at least two and, among the first two letters, *exactly* one has a bar.

The notations $D, \nabla, \bar{\nabla}, \parallel, \parallel$, were introduced in section 4.

Remark 6.1. Since covariant differentiation (with respect to g) and contraction with respect to g' *do not* commute, we observe that, for instance, the difference (recall $g' = g + \nabla\bar{\nabla}\phi$)

$$(3) \quad \phi_{aa'ab} - (\phi_{aa'\alpha})_b \equiv \phi_{ac\alpha} \phi_{a'c'b}$$

does not vanish.

7. HIGHER ORDER A PRIORI ESTIMATES: GENERALITIES

We want to prove by induction,

PROPOSITION 7.1. *Given $n \geq 4$, a sequence $(K_i), i \in \mathbb{N}$, and a finite sequence C_0, \dots, C_{n-1} , there exists C_n such that:*

$$\begin{aligned} \|\phi\| &\leq C_0, \quad \forall i = 0, \dots, n-3, \quad \|D^i \nabla \bar{\nabla} \phi\| \leq C_{i+2} \\ \text{and } \forall i \in \mathbb{N}, \quad \|D^i P_\lambda(\phi)\| &\leq K_i, \end{aligned}$$

implies

$$\|D^{n-2} \nabla \bar{\nabla} \phi\| \leq C_n.$$

Actually one needs $\|D^i P_\lambda(\varphi)\| \leq K_i$ only for $0 \leq i \leq n$, hence C_n depends only upon $(C_0, \dots, C_{n-1}, K_0, \dots, K_n)$.

Hereafter, by "a constant", we will mean a constant which depends only upon the given constants $(C_0, \dots, C_{n-1}, K_0, \dots, K_n)$.

Let us explain a further convention.

Convention 7.2. We will have to consider sums of tensors obtained via contractions of tensor polynomials in the variables $(g')^{-1}, \nabla \bar{\nabla} \varphi, \dots, D^i \nabla \bar{\nabla} \varphi, \dots$. The present convention helps describing the variables occurring in (still) uncontrolled expressions.

First of all, given $\varphi \in A_\lambda$ and an integer $n \geq 3$, we denote by E_{n-1} the (finite dimensional complex) vector space generated by all contracted tensor polynomials, with degree of homogeneity at most $2n$, in the variables

$$(g')^{-1}, \nabla \bar{\nabla} \varphi, D \nabla \bar{\nabla} \varphi, \dots, D^{n-3} \nabla \bar{\nabla} \varphi, D^i P_\lambda(\varphi), \quad i = 0, \dots, n.$$

In order to prove 7.1, we will compute *modulo* E_{n-1} .

Given integers p, \dots, s , all of them $\geq n$, we will say that *mod.* E_{n-1} a tensor T is "of the form $T_{p, \dots, s}$ ", whenever *mod.* E_{n-1} it is a sum of contractions of tensors

$$A \otimes D^{p-2} \nabla \bar{\nabla} \varphi \otimes \dots \otimes D^{s-2} \nabla \bar{\nabla} \varphi,$$

where the A 's are in E_{n-1} .

Furthermore for $s \geq n$, under the assumptions of 7.1, we will say that a scalar term $T_{s,s}$ is *coercive*, if for any other term of the form T'_s (resp. $T''_{s,s}$) there exists a constant C such that:

$$|T'_s| \leq C(T_{s,s})^{\frac{1}{2}} \quad (\text{resp. } |T''_{s,s}| \leq C T_{s,s}).$$

We present now three lemmas which illustrate the previous convention.

LEMMA 7.3. *Given integers $s \geq n \geq 3$, the covariant derivative (in metric g) of a term of the form $T_s \bmod. E_{n-1}$, is of the form $(T_{s+1} + T_s) \bmod. E_n$.*

Proof. This is just because the derivative $D[(g')^{-1}]$ is a contracted tensor polynomial (of degree 3) in $(g')^{-1}$ and $D \nabla \bar{\nabla} \varphi$.

LEMMA 7.4. *If α and β are two distinct mixed multi-indices of length $(n+2)$ obtained from each other by permutation, then the difference of covariant derivatives $(\varphi_\alpha - \varphi_\beta)$ is of the form $T_n \bmod. E_{n-1}$.*

Proof. On the Kähler manifold (X, g) , commuting two consecutive covariant derivatives yields curvature terms only if the couple of derivatives concerned is *mixed* (for general commutation rules on Riemannian manifolds see e.g. [21], exposé XI, proposition 3.2). If so, say k and \bar{l} are the permuted indices, the result will involve

$$R_{p\bar{k}l}^q \quad (\text{curvature tensor of } g)$$

with p and q of the same type. Explicitely:

$$\varphi_{\lambda k \bar{l} \mu} - \varphi_{\lambda \bar{l} k \mu} = \sum_p R_{p \bar{q} k \bar{l}} \varphi_{\nu q \tau}$$

for all p, ν, τ , such that $\nu p \tau \equiv \lambda \mu$. Hence the types of all the remaining non-permuted covariant derivatives $\varphi_{\nu q \tau}$ are *identically preserved*. In particular if γ and δ denote two multi-indices of length n obtained from each other by permutation, necessarily

$$(\varphi_{i \bar{j} \gamma} - \varphi_{i \bar{j} \delta}) \text{ is of the form } T_n \text{ mod. } E_{n-1},$$

since two *mixed* derivatives will keep bearing in first place on φ in the process of permutation.

The proof of lemma 7.4 is therefore reduced to the following two cases for the multi-indices α and β :

$$\begin{aligned} \text{either } \alpha &= i \bar{j} k \lambda, \quad \beta = k \bar{j} i \lambda, \quad |\lambda| = n - 1, \\ \text{or } \alpha &= i \bar{j} k \bar{l} \mu, \quad \beta = k \bar{l} i \bar{j} \mu, \quad |\mu| = n - 2. \end{aligned}$$

In the first case, one has identically on a Kähler manifold:

$$\varphi_{\alpha} - \varphi_{\beta} \equiv 0.$$

In the second case, the same reasoning as above holds for $(\varphi_{\alpha} - \varphi_{\beta})$ since it can be written as

$$(\varphi_{i \bar{j} k \bar{l} \mu} - \varphi_{i k \bar{j} \bar{l} \mu}) + (\varphi_{k i \bar{l} \bar{j} \mu} - \varphi_{k \bar{l} i \bar{j} \mu}),$$

each of these two commutations being clearly of the form $T_n \text{ mod. } E_{n-1}$.
Q.E.D.

Remark 7.5. The fact that commutation formulae involve only *mixed* derivatives was already a crucial detail in the proofs of the second and third order *a priori* estimates.

LEMMA 7.6. The tensor $\varphi_{a a' \alpha}$ where α is a mixed multi-index of length n is, mod. E_{n-1} , of the form:

$$\begin{array}{ll}
T_{3,3} + T_2 & \text{when } n = 2, \\
T_{4,3} + T_{3,3,3} + T_3 & \text{when } n = 3, \\
T_5 + T_{4,4} + T_4 & \text{when } n = 4, \\
T_{n+1} + T_n & \text{when } n \geq 5.
\end{array}$$

Proof. The cases $n = 2, 3, 4, 5$, must be checked bare-handed. There is no difficulty. Then, for $n \geq 5$, one can proceed by induction on n . Indeed assume,

$$\varphi_{aa'\alpha} = T_{n+1} + T_n \bmod. E_{n-1}, \quad \text{for some } n = |\alpha| \geq 5.$$

Recall formula (3) and lemma 7.3; differentiating once the above equality yields

$$\varphi_{aa'\alpha b} = (T_{n+1} + T_n)_b + \varphi_{ac\alpha} \varphi_{a'c'b} = T_{n+2} + T_{n+1} \bmod. E_n,$$

since $|ac\alpha| = n + 2$. The same is true with \bar{b} instead of b . Q.E.D.

Remark 7.7. The preceding lemma offers a perspective which brings some light on the type of difficulties to be expected for carrying out *a priori* estimates of each order. In particular, one may anticipate that a special step should be required for $n = 4$ (in order to kill the effect of the term $T_{4,4}$) and that the same (simpler) procedure should then apply, arguing by iteration, for any $n \geq 5$.

Notice also that the hardest case appears to be $n = 3$. Indeed, following Calabi [8] one must guess the very special *coercive* functional [1] [24]

$$S_{3,3} = \varphi_{ab'c} \varphi_{a'bc'},$$

perform a careful calculation of $\Delta'(S_{3,3})$ and use either the Maximum Principle [24] or a recurrence on $L^p(dX_{g'})$ norms of $S_{3,3}$ [1]. The *approximate* tensor calculus which we may conveniently use hereafter would not be effective for the case $n = 3$.

8. A PRIORI ESTIMATES OF ORDER FOUR

In order to prove 7.1 with $n = 4$, we consider the functional:

$$S_{4,4} = \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}} \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}} + \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}} \varphi_{\bar{a}\bar{b}\bar{c}\bar{d}}.$$

It is enough to estimate $S_{4,4}$ since it is *coercive*. Let us compute $-\Delta'(S_{4,4})$. One readily obtains:

$$-\Delta'(S_{4,4}) = T_{6,4} + T_{5,5} \quad (\bmod. E_3),$$