

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 34 (1988)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ABOUT THE PROOFS OF CALABI'S CONJECTURES ON COMPACT KÄHLER MANIFOLDS  
**Autor:** Delanoë, Ph. / Hirschowitz, A.  
**Kapitel:** 2. A Topological Lemma  
**DOI:** <https://doi.org/10.5169/seals-56591>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 30.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

from  $\{u \in C^\infty(X), \int u dX_g = 0\}$  to  $\{v \in C^\infty(X), \int v dX_{g'} = 0\}$  ( $dX_{g'}$  denotes the volume form in the metric  $g'$ ) when  $\lambda = 0$ .

For completeness, let us indicate how, for instance theorem 0.2, can be reduced to equation (1) with  $\lambda = 0$ . It is quite straightforward. First of all we are given a cohomology class  $c \in H^2(X, \mathbf{R})$  such that there exists a Kähler form  $\omega$  in  $c$ ; let  $\rho$  be the Ricci form of  $\omega$ :  $\rho \in C_1(X)$ , the first Chern class of  $X$ .

Then we are given  $\gamma \in C_1(X)$  and hence  $f \in C^\infty(X)$  a real function (defined up to an additive constant), which measures the deviation for  $\omega$  from satisfying 0.2:

$$\gamma - \rho = \sqrt{-1} \partial \bar{\partial} f.$$

Now we look for another Kähler form  $\omega' \in c$ , i.e. we look for a smooth real function  $\varphi$  (also defined up to an additive constant), where

$$\omega' - \omega = \sqrt{-1} \partial \bar{\partial} \varphi$$

such that the Ricci form  $\rho'$  of  $\omega'$  coincides with  $\gamma$ .

In other words, we want  $\varphi$  to satisfy

$$\rho' - \rho \equiv \sqrt{-1} \partial \bar{\partial} \varphi,$$

or equivalently, if  $g$  and  $g'$  are the Kähler metrics respectively associated with  $\omega$  and  $\omega'$ ,

$$\partial \bar{\partial} \{-\text{Log det}(g'g^{-1})\} \equiv \partial \bar{\partial} \varphi$$

which immediately yields equation (1) with  $\lambda = 0$ :

$$-\text{Log det}(g'g^{-1}) = f,$$

since anyway  $f$  is only defined up to an additive constant.

As  $\omega$  and  $\omega'$  are cohomologous and closed, so are the corresponding volume forms, therefore  $X$  has same volume measured with the metrics  $g$  and  $g'$ ; this defines completely  $f$ , subject to the natural constraint mentioned above.

## 2. A TOPOLOGICAL LEMMA

In our setting, the continuity method becomes a "surjectivity method" since it is based on the following

LEMMA 2.1. Let  $A, B$  be metric spaces, with  $A \neq \emptyset$  and  $B$  connected. Let  $P: A \rightarrow B$  be a continuous map. Assume:

- (i)  $P$  is open,
- (ii)  $P$  is proper, that is, for any compact subset  $K$  in  $B$ ,  $P^{-1}(K)$  is compact. Then  $P$  is surjective.

*Proof.* We only need to prove that  $P(A)$  is closed. Let  $b_0$  be a point in  $\overline{P(A)}$ . Since  $B$  is a metric space, there exists a sequence  $(b_i)_{i>0}$  in  $P(A)$  converging to  $b_0$ . The subset  $K = \{b_0, b_1, b_2, \dots\}$  is compact, hence so is  $PP^{-1}(K)$ . The latter contains  $b_1, \dots, b_i, \dots$ , hence  $b_0$ , and it is obviously contained in  $P(A)$ . Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

### 3. LOCAL INVERSION

THEOREM 3.1. Let  $X$  be a smooth compact manifold,  $V$  and  $W$  smooth vector bundles on  $X$ ,  $U$  an open set in  $C^\infty(X, V)$ , and  $P: U \rightarrow C^\infty(X, W)$ , a smooth nonlinear elliptic partial differential operator. Let  $A$  and  $B$  be LCFC submanifolds of  $U$  and of  $C^\infty(X, W)$  respectively, such that the restriction  $P_A$  of  $P$  to  $A$ , sends  $A$  into  $B$ . Then the Jacobian criterion holds for  $P_A$ , namely, if the derivative of  $P_A: A \rightarrow B$  is invertible at  $\varphi_0 \in A$ , then  $P_A$  is a local diffeomorphism near  $\varphi_0$ .

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the classical (Banach) inverse function theorem combined with *elliptic regularity*.

*Remark 3.2.* The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

### 4. PROPERNESS

In view of (2), theorem 3.1 implies that  $P_\lambda$  is open. We want to apply lemma 2.1 in order to prove that  $P_\lambda$  is surjective from  $A_\lambda$  to  $B_\lambda$ . Since  $P_\lambda(A_\lambda) \neq \emptyset$  (it contains 0), and since  $B_\lambda$  is connected, this amounts to proving that  $P_\lambda$  is *proper*. Let us explain why *a priori* estimates imply properness.

Concerning subsets in  $A_\lambda$  we have