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3.3. LEMMA. Let  $R$  be a local ring and  $F$  be the associated field. Let  $f: C_1 \rightarrow C_0$  be a  $R$ -homomorphism of finitely generated free  $R$ -modules and let  $\bar{f}: F \otimes_R C_1 \rightarrow F \otimes_R C_0$  be the induced  $F$ -homomorphism. If  $\text{rk } f = \text{rk } \bar{f}$  then with respect to some bases in  $C_1, C_0$  the homomorphism  $f$  is presented by the matrix  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$  where  $E$  is the unit matrix of order  $\text{rk } f$ .

*Proof.* Since  $F$  is a field we can choose bases  $d_0, d_1$  respectively in  $F \otimes_R C_0, F \otimes_R C_1$  so that the matrix of  $\bar{f}$  regarding these bases has the form  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ . Let  $\mathcal{D}_i$  be a lifting of  $d_i$  to  $C_i, i = 1, 2$ . Here  $\mathcal{D}_i$  is a sequence of  $\text{rg } C_i$  elements of  $C_i$ . In view of Nakayama's lemma  $\mathcal{D}_i$  generate  $C_i$ . This implies that  $\mathcal{D}_i$  generates the  $(\text{rg } C_i)$ -dimensional vector space  $Q(R) \otimes_R C_i$  over the field  $Q(R)$ . Therefore, the elements of the sequence  $\mathcal{D}_i$  are linearly independent over  $Q(R)$  and, hence, over  $R$ . Thus  $\mathcal{D}_i$  is a basis of  $C_i$  for  $i = 0, 1$ . The matrix of  $f$  with respect to bases  $\mathcal{D}_0, \mathcal{D}_1$  has the form  $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$  where  $U, X, Y, Z$  are matrices over the maximal ideal  $u$  of  $R$ . Note that  $\det(E+U) = 1 \pmod{u}$ . Since all elements of  $R \setminus U$  are invertible in  $R$  the square matrix  $E+U$  is invertible over  $R$ . Therefore we can choose bases in  $C_0, C_1$  so that the corresponding matrix of  $f$  equals  $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$ . Since  $\text{rk } f = \text{rk } \bar{f} = \text{rk } E, Y' = 0$ .

3.4. LEMMA. Let  $R$  be a local ring and  $F$  be the associated field. Let  $C = (\cdots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \cdots)$  be a finitely generated free chain complex over  $R$ . Let  $C'$  be the chain  $F$ -complex  $F \otimes_R C$ . Let  $\partial_i, \partial'_i$  be the boundary homomorphisms  $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$ . If  $\text{rk}_R H_i(C) = \text{rk}_F H_i(C')$  for some  $i$  then:  $H_i(C), \text{Im } \partial_{i+1}, \text{Im } \partial_i$  are free  $R$ -modules and  $C_i = \text{Im } \partial_{i+1} \oplus H_i(C) \oplus \text{Im } \partial_i$ ; the projection  $C \rightarrow C'$  induces  $F$ -isomorphisms  $F \otimes_R H_i(C) \rightarrow H_i(C'), F \otimes_R \text{Im } \partial_j \rightarrow \text{Im } \partial'_j$  with  $j = i, i+1$ .

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

#### § 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by  $Q_n$  the fraction field of the ring  $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ . Denote by  $Q_n^0$  the subring of  $Q_n$  which consists of rational functions  $fg^{-1}$  with  $f, g \in \Lambda_n$  and  $g \notin (t_n - 1)\Lambda_n$  (so that

$g(t_1, \dots, t_{n-1}, 1) \neq 0$ ). The homomorphism  $f \mapsto f(t_1, \dots, t_{n-1}, 1): \Lambda_n \rightarrow \Lambda_{n-1}$  uniquely extends to a ring homomorphism  $Q_n^0 \rightarrow Q_{n-1}$  which is denoted by  $\varphi$ .

Denote by  $X$  the exterior of  $K$  and by  $Y$  the exterior of  $L$ .

We shall prove the following two statements.

(4.1.1).  $\varphi(\Delta(K)) = \Delta(K)(t_1, \dots, t_{n-1}, 1)$  divides  $\Delta(L)$  in  $\Lambda_{n-1}$ .

(4.1.2). There exists a representative  $\omega$  of the torsion  $\omega(X) \subset Q_n$  such that  $(t_n - 1)\omega \in Q_n^0$  and  $\varphi((t_n - 1)\omega)$  represents  $\omega(Y) \subset Q_{n-1}$ .

Let us show first that these two statements imply the Theorem. Let  $\omega$  be the element of  $Q_n$  produced by (4.1.2). Put  $\pi = \prod_{i=1}^{n-1} (t_i - 1)$ . According to the results formulated in Sec. 2.2 the product  $(t_n - 1)\pi \cdot \Delta(K)$  represents  $\omega(X)$ . Thus

$$\omega \doteq \frac{f\bar{f}}{g\bar{g}}(t_n - 1)\pi\Delta(K)$$

where  $f, g \in \Lambda_n \setminus 0$ . We may assume that  $f\bar{f}$  and  $g\bar{g}$  are relatively prime. If  $t_n - 1$  does not divide  $g$  then  $\omega \in Q_n^0$  and  $\varphi((t_n - 1)\omega) = 0$  which contradicts to the inclusion  $\varphi((t_n - 1)\omega) \in \omega(Y)$ . Thus  $g = (t_n - 1)h$  with  $h \in \Lambda_n$ . In view of (4.1.1),  $\varphi(\Delta(K)) \neq 0$ , i.e.  $t_n - 1$  does not divide  $\Delta(K)$ . If  $\varphi(h) = 0$  then  $(t_n - 1)^2$  divides  $g$  which obviously contradicts the inclusion  $(t_n - 1)\omega \in Q_n^0$ . Thus  $\varphi(h) \neq 0$ . We have

$$h\bar{h}(t_n - 1)\omega \doteq f\bar{f}\pi\Delta(K).$$

Since  $\varphi(h\bar{h}(t_n - 1)\omega) \neq 0$  we have  $\varphi(f) \neq 0$ . This implies that  $\pi \cdot \varphi(\Delta(K)) \doteq q\bar{q}\varphi((t_n - 1)\omega)$  where  $q = \varphi(h)/\varphi(f)$ . Thus  $\pi\varphi(\Delta(K))$  represents  $\omega(Y)$ . Since  $\pi\Delta(L) \in \omega(Y)$  we have

$$\varphi(\Delta(K))\lambda\bar{\lambda} = \Delta(L)\mu\bar{\mu}$$

with non-zero  $\lambda, \mu \in \Lambda_{n-1}$ . We may assume that  $\lambda\bar{\lambda}$  and  $\mu\bar{\mu}$  are relatively prime. Since  $\varphi(\Delta(K))$  divides  $\Delta(L)$  we immediately obtain  $\mu\bar{\mu} = 1$ . Thus,  $\Delta(L) = \varphi(\Delta(K))\lambda\bar{\lambda}$ .

Let us prove (4.1.1) and (4.1.2). We may assume that  $X \subset Y$  and that  $Y \setminus X$  is the interior of the regular neighborhood  $U \subset Y$  of the  $n$ -th component of  $K$  in  $Y$ . Let  $p: \tilde{X} \rightarrow X$  and  $q: \tilde{Y} \rightarrow Y$  be the maximal abelian coverings with the groups of covering transformations respectively  $H_1(X) \approx \mathbf{Z}^n$  (generators  $t_1, \dots, t_n$ ) and  $H_1(Y) \approx \mathbf{Z}^{n-1}$  (generators  $t_1, \dots, t_{n-1}$ ). It is clear that  $p$  is the composition of an infinite cyclic covering  $\tilde{X} \rightarrow q^{-1}(X)$  and the covering  $q: q^{-1}(X) \rightarrow X$ .

Fix a  $C^1$ -triangulation of  $Y$  so that  $X$  and  $U$  are simplicial subcomplexes of  $Y$ . Fix also the induced equivariant triangulations in  $\tilde{X}$  and  $\tilde{Y}$ .

The ring  $\Lambda_{n-1}$  determines via the natural homomorphism  $\mathbf{Z}[\pi_1(Y)] \rightarrow \mathbf{Z}[H_1 Y] = \Lambda_{n-1}$  a system of local coefficients on  $Y$  which we denote by the same symbol  $\Lambda_{n-1}$ . According to definitions, for any simplicial subsets  $A \supset B$  of  $Y$  the  $\Lambda_{n-1}$ -module  $H_*(A, B; \Lambda_{n-1})$  equals  $H_*(C(q^{-1}(A), q^{-1}(B); \mathbf{Z}))$ . Here the simplicial chain complex  $C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$  is a finitely generated free  $\Lambda_{n-1}$ -complex. Analogously  $\Lambda_n$  defines a system of local coefficients on  $X$  and for simplicial subsets  $A \supset B$  of  $X$  the  $\Lambda_n$ -module  $H_*(A, B; \Lambda_n)$  equals  $H_*(C(p^{-1}(A), p^{-1}(B); \mathbf{Z}))$ . Note that

$$\Lambda_{n-1} \otimes_{\Lambda_n} C_*(p^{-1}(A), p^{-1}(B); \mathbf{Z}) = C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$$

where  $\Lambda_n$  acts on  $\Lambda_{n-1}$  via  $\phi$ .

*Claim 1.* For  $i \neq 1, m-1$ ,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = 0.$$

For  $i = 1, m-1$ ,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = n-1; \quad \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = n-2.$$

*Proof of Claim 1.* We shall compute the rank of  $H_i(X; \Lambda_n)$ ; modules  $H_i(X; \Lambda_{n-1})$  and  $H_i(Y; \Lambda_{n-1})$  can be treated similarly.

Denote by  $V$  a wedge of  $n$  circles in  $X$  such that the inclusion homomorphism  $H_1(V; \mathbf{Z}) \rightarrow H_1(X; \mathbf{Z}) = \mathbf{Z}^n$  is bijective. Then  $H_i(X, V, \mathbf{Z}) = 0$  for  $i \leq m-2$ . Therefore an application of Lemma 3.2(i) to complexes  $C_*(\tilde{X}, p^{-1}(V); \mathbf{Z})$  and  $C_*(X, V; \mathbf{Z})$  gives that  $\mathrm{rk}_{\Lambda_n} H_i(X, V; \Lambda_n) = 0$  for  $i \leq m-2$ . This implies that  $\mathrm{rk} H_i(X; \Lambda_n) = \mathrm{rk} H_i(V; \Lambda_n)$  for  $i \leq m-3$  and that  $\mathrm{rk} H_{m-2}(X; \Lambda_n) \leq \mathrm{rk} H_{m-2}(V; \Lambda_n)$ . The rank of  $H_i(V; \Lambda_n)$  can be computed directly: It is equal to 0 if  $i \neq 1$  and to  $n-1$  if  $i = 1$ . Thus the rank of  $H_i(X; \Lambda_n)$  equals 0 if  $i \neq 1, m-1$  and equals  $n-1$  if  $i = 1$ . The equality  $\mathrm{rk} H_{m-1}(X; \Lambda_n) = n-1$  follows from duality or from the equalities

$$\sum_{i=0}^m (-1)^i \mathrm{rk} H_i(X; \Lambda_n) = \chi(X) = 0.$$

*Claim 2.* The exact homology sequence of  $(Y, X)$  with coefficients in  $\Lambda_{n-1}$  splits into short exact sequences

$$\begin{aligned}
0 &\rightarrow H_m(Y, X; \Lambda_{n-1}) \rightarrow H_{m-1}(X; \Lambda_{n-1}) \rightarrow H_{m-1}(Y; \Lambda_{n-1}) \rightarrow 0, \\
0 &\rightarrow H_i(X; \Lambda_{n-1}) \xrightarrow{\approx} H_i(Y; \Lambda_{n-1}) \rightarrow 0, \quad (i \neq 1, m-1) \\
0 &\rightarrow H_2(Y, X; \Lambda_{n-1}) \xrightarrow{\partial_1} H_1(X; \Lambda_{n-1}) \rightarrow H_1(Y; \Lambda_{n-1}) \rightarrow 0.
\end{aligned}$$

*Proof of Claim 2.* Clearly,  $H_i(Y, X; \Lambda_{n-1}) = H_i(U, \partial U; \Lambda_{n-1}) = 0$  for  $i \neq 2, m$ . Therefore the only thing to prove is the injectivity of  $\partial_1$ . According to Claim 1  $\text{rk } H_1(X; \Lambda_{n-1}) = n - 1$  and  $\text{rk } H_1(Y; \Lambda_{n-1}) = n - 2$ . Since  $H_2(Y, X; \Lambda_{n-1}) = \Lambda_{n-1}$  we see that  $\partial_1$  is injective.

*Proof of (4.1.1).* In view of the equalities  $\text{rg } H_i(X; \Lambda_n) = \text{rg } H_i(X; \Lambda_{n-1})$ ,  $i = 0, 1, \dots$  we may apply Lemma 3.2 (iii) to the chain complexes  $C_*(\tilde{X}; \mathbf{Z})$  and  $C_*(q^{-1}(X); \mathbf{Z})$  respectively over  $\Lambda_n$  and  $\Lambda_{n-1}$ . Since  $m - 1 > r > 1$  Claims 1, 2 show that  $H_r(X; \Lambda_n)$  and  $H_r(X; \Lambda_{n-1})$  are torsion modules respectively over  $\Lambda_n$  and  $\Lambda_{n-1}$  and  $H_r(X, \Lambda_{n-1}) = H_r(Y; \Lambda_{n-1})$ . By definition  $\Delta(K) = \text{ord } H_r(X; \Lambda_n)$  and  $\Delta(L) = \text{ord } H_r(Y; \Lambda_{n-1}) = \text{ord } H_r(X; \Lambda_{n-1})$ . Lemma 3.2 (iii) directly implies that  $\phi(\Delta(K))$  divides  $\Delta(L)$ .

It remains to prove Statement (4.1.2) which is, of course, the core of Theorem 1. For simplicial subsets  $A \supset B$  of  $Y$  we shall denote by  $C(A, B)$  the (simplicial) chain  $Q_{n-1}$ -complex  $Q_{n-1} \otimes_{\Lambda_{n-1}} C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$ . Clearly

$$H_i(A, B; Q_{n-1}) = H_i(C(A, B)) = Q_{n-1} \otimes_{\Lambda_{n-1}} H_i(A, B; \Lambda_{n-1}).$$

Consider the short exact sequence of chain  $Q_{n-1}$ -complexes

$$(5) \quad 0 \rightarrow C(X) \rightarrow C(Y) \rightarrow C(Y, X) \rightarrow 0.$$

Provide the homology modules of complexes  $C(X)$ ,  $C(Y)$ ,  $C(Y, X)$  with bases as follows. It is evident that  $H_i(C(Y, X)) = 0$  for  $i \neq 2, m$  and

$$H_i(C(Y, X)) = H_i(C(U, \partial U)) = H_i(U, \partial U; Q_{n-1}) = Q_{n-1}$$

for  $i = 2, m$ . Fix a lifting  $\tilde{U} \subset \tilde{Y}$  of  $U \approx S^{m-2} \times D^2$ . Fix in  $H_m(C(Y, X))$  the generator  $[\tilde{U}, \partial \tilde{U}]$ . Fix in  $H_2(C(Y, X))$  the generator  $[\Delta, \partial \Delta]$  where  $\Delta$  is the meridional disk of  $\tilde{U}$ .

It follows from Claim 1 that  $H_i(C(X)) = H_i(C(Y)) = 0$  for  $i \neq 1, m - 1$ . Fix an arbitrary basis  $f$  in the  $(n-2)$ -dimensional vector  $Q_{n-1}$ -space  $H_1(Y; Q_{n-1})$ . Fix the dual basis  $g$  in  $H_{m-1}(Y; Q_{n-1})$ . It follows from Claim 2 that inclusion homomorphisms  $H_i(C(X)) \rightarrow H_i(C(Y))$  are surjective for all  $i$ . Let  $F$  and  $G$  be sequences of  $n - 2$  vectors in  $H_1(C(X))$  and in  $H_{m-1}(C(X))$  whose images under these inclusion homomorphisms are equal respectively to  $f$  and  $g$ . Claim 2 implies that  $[\partial \tilde{U}], G$  is a basis in  $H_{m-1}(C(X))$  and

$[\partial\Delta]$ ,  $F$  is a basis in  $H_1(C(X))$ . Now all homology modules of complexes  $C(X)$ ,  $C(Y)$ ,  $C(Y, X)$  are provided with bases.

Provide the modules of  $C(X)$ ,  $C(Y)$ ,  $C(Y, X)$  with natural bases (see Sec. 2.2). We may choose these bases to be compatible in the sense of Lemma 2.1.1. According to this Lemma

$$\tau(C(Y)) = \pm \tau(C(X))\tau(C(Y, X))\tau(\mathcal{H})$$

where  $\mathcal{H}$  is the homology sequence associated with the exact sequence (5). It is evident that  $\tau(\mathcal{H}) = \pm 1$ . It is easy to verify that  $\tau(C(Y, X)) = \tau(C(U, \partial U)) = \pm 1$ . (Indeed, the pair  $(U, \partial U)$  has a cell structure such that  $\text{Int } U$  contains 2 open cells; the meridional disc and its complement; for such cell structure the equality  $\tau(C(U, \partial U)) = \pm 1$  is evident. The case of an arbitrary cell structure (or triangulation) follows from the invariance of torsion under cell subdivision). Thus  $\tau(C(Y)) = \pm \tau(C(X))$ . Note that  $\tau(C(Y))$  represents  $\omega(Y)$ . Therefore  $\tau(C(X))$  also represents  $\omega(Y)$ .

Consider the chain complex

$$C = Q_n^0 \otimes_{\Lambda_n} C_*(\tilde{X}; \mathbf{Z}).$$

Note that  $Q_n^0$  is a local ring with the maximal ideal  $(t_n - 1)Q_n^0$  and associated field  $Q_{n-1}$ . Clearly,  $Q_{n-1} \otimes_{Q_n^0} C = C(X)$ . The natural bases in chain modules of  $C(X)$  lift to natural bases in chain modules of  $C$ . Claim 1 implies that for all  $i \geq 0$

$$\text{rk}_{Q_n^0} H_i(C) = \text{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \text{rk}_{Q_{n-1}} H_i(C(X)).$$

Therefore we may apply Lemma 3.4 to complexes  $C$ ,  $C(X)$ . This lemma shows that:  $H_i(C) = H_i(C(X)) = 0$  for  $i \neq 1, m - 1$ ; the basis  $[\partial\Delta]$ ,  $F$  in  $H_1(C(X))$  lifts to a basis, say,  $f_0, f_1, \dots, f_{n-2}$  in  $H_1(C)$ ; the basis  $[\partial\tilde{U}]$ ,  $G$  in  $H_{m-1}(C(X))$  lifts to a basis, say,  $g_0, g_1, \dots, g_{n-2}$  in  $H_{m-1}(C)$ ; the submodules of cycles and boundaries of  $C$  are free in all dimensions. Thus we may apply the constructions of Sec. 2.1 to  $C$  which gives rise to a torsion  $\tau(C) \in Q_n^0$ . It follows directly from the formula (3) that  $\phi(\tau(C)) = \tau(C(X))$ . Thus  $\phi(\tau(C))$  represents  $\omega(Y)$ .

Let  $v$  be the matrix of the semi-linear intersection pairing

$$\langle \ , \ \rangle : H_1(X; Q_n^0) \times H_{m-1}(X; Q_n^0) \rightarrow Q_n^0$$

with respect to bases  $f_0, f_1, \dots, f_{n-2}$  and  $g_0, g_1, \dots, g_{n-2}$ . (Here  $H_i(X; Q_n^0) = H_i(C)$ ). It is clear that  $\tau(C) (\det v)^{-1}$  represents  $\omega(X)$ . Put  $\omega = \tau(C) (\det v)^{-1}$ . We shall prove that

$$(6) \quad \det v = \pm (t_n - 1) + (t_n - 1)^2 a$$

where  $a \in Q_n^0$ . Then  $(t_n - 1)\omega \in Q_n^0$  and

$$\varphi((t_n - 1)\omega) = \varphi(\tau(C)[\pm 1 + (t_n - 1)a]^{-1}) = \pm \varphi(\tau(C)) \in \omega(Y).$$

This would complete the proof of (4.1.2).

It is obvious that

$$v = \begin{bmatrix} \langle f_0, g_0 \rangle & (t_n - 1)\alpha \\ (t_n - 1)\beta & E + (t_n - 1)\gamma \end{bmatrix}$$

where  $\alpha, \beta, \gamma$  are respectively a  $(n-2)$ -row,  $(n-2)$ -column and  $(n-2) \times (n-2)$ -matrix over  $Q_n^0$ . It turns out that

$$(7) \quad \langle f_0, g_0 \rangle = \pm (t_n - 1) + (t_n - 1)^2 b$$

with  $b \in Q_n^0$ . This immediately implies (6).

I shall prove (7) for a special choice of  $f_0$  which is sufficient for our aims. Let  $\theta: [0, 1] \rightarrow \partial\tilde{X}$  be a path whose projection to  $\tilde{Y}$  is a loop parametrizing  $\partial\Delta \subset \partial\tilde{U}$ . Let  $\eta: [0, 1] \rightarrow \tilde{X}$  be a path such that  $\eta(0) = \theta(0)$  and  $\eta(1) = t_1 \cdot \theta(0)$ . Consider the singular chain  $\mathfrak{g} = \theta - t_1\theta + t_n\eta - \eta$ . It is easy to check up that  $\mathfrak{g}$  is a cycle in  $\tilde{X}$  and that its homology class  $[\mathfrak{g}] \in H_1(C)$  projects to  $(1 - t_1)[\partial\Delta] \in H_1(C(X))$ . Put  $f_0 = (1 - t_1)^{-1}[\mathfrak{g}]$ . Then  $\langle f_0, g_0 \rangle = (1 - t_1)^{-1}\langle [\mathfrak{g}], g_0 \rangle = (1 - t_1)^{-1}(t_n - 1)\langle \eta, g_0 \rangle$  where in the right part the brackets  $\langle \cdot, \cdot \rangle$  denote the intersection pairing

$$H_1(X, \partial X; Q_n^0) \times H_{m-1}(X; Q_n^0) \rightarrow Q_n^0.$$

The image of  $\langle \eta, g_0 \rangle$  under  $\varphi: Q_n^0 \rightarrow Q_{n-1}$  can be computed using the analogous pairing

$$H_1(X, \partial X; Q_{n-1}) \times H_{m-1}(X; Q_{n-1}) \rightarrow Q_{n-1}.$$

Namely,  $\varphi(\langle \eta, g_0 \rangle) = \pm (t_1 - 1)$ . Thus  $\langle \eta, g_0 \rangle = \pm (t_1 - 1) + (t_n - 1)c$  with  $c \in Q_n^0$ . Therefore  $\langle f_0, g_0 \rangle = \pm (t_n - 1) + (t_n - 1)^2 b$  where  $b = (1 - t_1)^{-1}c$ . This implies (7).

**4.2. Proof of Theorem 2.** We may assume that  $\Delta_{u-1}(L) \neq 0$  and  $l_1 = l_2 = \dots = l_{n-1} = 0$ . Then the  $n$ -th component of  $K$  lifts to the maximal abelian covering of the exterior  $Y$  of  $L$ . The remaining part of the proof is analogous to the proof of Theorem 1. Note, however, the necessary changes. In Claim 1 for  $i = 1, 2$

$$\text{rk}_{\Lambda_n} H_i(X; \Delta_n) = \text{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = u - 1; \text{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = u - 2.$$

In the proof of (4.1.1) one should take into account that  $\text{Tors}_{\Lambda_{n-1}} H_1(X; \Lambda_{n-1})$  injects into  $\text{Tors}_{\Lambda_{n-1}} H_1(Y; \Lambda_{n-1})$  and thus the order of the first of these 2 modules divides the order of the second one.

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