3.3. LEMMA. *Let R be a local ring and F be the associated field. Let $f: C_1 \to C_0$ be a R-homomorphism of finitely generated free R-modules and let $\bar{f}: F \otimes_R C_1 \to F \otimes_R C_0$ be the induced F-homomorphism. If* $\operatorname{rk} f = \operatorname{rk} \bar{f}$ *then with respect to some bases in C_1, C_0 the homomorphism f is presented by the matrix* $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ *where E is the unit matrix of order* $\operatorname{rk} f$.

Proof. Since F is a field we can choose bases d_0, d_1 respectively in $F \otimes_k C_0$, $F \otimes_K C_1$ so that the matrix of $\bar{f}$ regarding these bases has the form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Let $\mathscr{D}_i$ be a lifting of d_i to $C_i, i = 1, 2$. Here $\mathscr{D}_i$ is a sequence of $\operatorname{rg} C_i$ elements of C_i. In view of Nakayama's lemma $\mathscr{D}_i$ generate C_i. This implies that $\mathscr{D}_i$ generates the $(\operatorname{rg} C_i)$-dimensional vector space $Q(R) \otimes_R C_i$ over the field $Q(R)$. Therefore, the elements of the sequence $\mathscr{D}_i$ are linearly independent over $Q(R)$ and, hence, over R. Thus $\mathscr{D}_i$ is a basis of C_i for $i = 0, 1$. The matrix of f with respect to bases $\mathscr{D}_0, \mathscr{D}_1$ has the form $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$ where U, X, Y, Z are matrices over the maximal ideal u of R. Note that $\det(E+U) = 1 \pmod u$. Since all elements of $R \backslash U$ are invertible in R the square matrix $E + U$ is invertible over R. Therefore we can choose bases in C_0, C_1 so that the corresponding matrix of f equals $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$. Since $\operatorname{rk} f = \operatorname{rk} \bar{f} = \operatorname{rk} E$, $Y' = 0$.

3.4. LEMMA. *Let R be a local ring and F be the associated field. Let $C = (\cdots \to C_{i+1} \to C_i \to \cdots)$ be a finitely generated free chain complex over R. Let C' be the chain F-complex $F \otimes_R C$. Let ∂_i, ∂'_i be the boundary homomorphisms $C_{i+1} \to C_i, C'_{i+1} \to C'_i$. If* $\operatorname{rk}_R H_i(C) = \operatorname{rk}_F H_i(C')$ *for some i then: $H_i(C), \operatorname{Im} \partial_{i+1}, \operatorname{Im} \partial_i$ are free R-modules and $C_i = \operatorname{Im} \partial_{i+1} \oplus H_i(C) \oplus \operatorname{Im} \partial_i$; the projection $C \to C'$ induces F-isomorphisms $F \otimes_R H_i(C) \to H_i(C'), F \otimes_R \operatorname{Im} \partial_j \to \operatorname{Im} \partial'_j$ with $j = i, i+1$.*

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

§ 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by Q_n the fraction field of the ring $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. Denote by Q_n^0 the subring of Q_n which consists of rational functions fg^{-1} with $f, g \in \Lambda_n$ and $g \notin (t_n - 1)\Lambda_n$ (so that

$g(t_1, \ldots, t_{n-1}, 1) \neq 0)$. The homomorphism $f \mapsto f(t_1, \ldots, t_{n-1}, 1): \Lambda_n \to \Lambda_{n-1}$ uniquely extends to a ring homomorphism $Q_n^0 \to Q_{n-1}$ which is denoted by φ.

Denote by X the exterior of K and by Y the exterior of L.

We shall prove the following two statements.

(4.1.1). $\varphi(\Delta(K)) = \Delta(K)(t_1, \ldots, t_{n-1}, 1)$ divides $\Delta(L)$ in Λ_{n-1}.

(4.1.2). There exists a representative ω of the torsion $\omega(X) \subset Q_n$ such that $(t_n - 1)\omega \in Q_n^0$ and $\varphi((t_n - 1)\omega)$ represents $\omega(Y) \subset Q_{n-1}$.

Let us show first that these two statements imply the Theorem. Let ω be the element of Q_n produced by (4.1.2). Put $\pi = \prod_{i=1}^{n-1} (t_i - 1)$. According to the results formulated in Sec. 2.2 the product $(t_n - 1)\pi \cdot \Delta(K)$ represents $\omega(X)$. Thus

$$\omega \doteq \frac{f\bar{f}}{g\bar{g}} (t_n - 1)\pi\Delta(K)$$

where $f, g \in \Lambda_n \backslash 0$. We may assume that $f\bar{f}$ and $g\bar{g}$ are relatively prime. If $t_n - 1$ does not divide g then $\omega \in Q_n^0$ and $\varphi((t_n - 1)\omega) = 0$ which contradicts to the inclusion $\varphi((t_n - 1)\omega) \in \omega(Y)$. Thus $g = (t_n - 1)h$ with $h \in \Lambda_n$. In view of (4.1.1), $\varphi(\Delta(K)) \neq 0$, i.e. $t_n - 1$ does not divide $\Delta(K)$. If $\varphi(h) = 0$ then $(t_n - 1)^2$ divides g which obviously contradicts the inclusion $(t_n - 1)\omega \in Q_n^0$. Thus $\varphi(h) \neq 0$. We have

$$h\bar{h}(t_n - 1)\omega \doteq f\bar{f}\,\pi\Delta(K).$$

Since $\varphi(h\bar{h}(t_n - 1)\omega) \neq 0$ we have $\varphi(f) \neq 0$. This implies that $\pi \cdot \varphi(\Delta(K)) \doteq q\bar{q}\,\varphi((t_n - 1)\omega)$ where $q = \varphi(h)/\varphi(f)$. Thus $\pi\varphi(\Delta(K))$ represents $\omega(Y)$. Since $\pi\Delta(L) \in \omega(Y)$ we have

$$\varphi(\Delta(K))\lambda\bar{\lambda} = \Delta(L)\mu\bar{\mu}$$

with non-zero $\lambda, \mu \in \Lambda_{n-1}$. We may assume that $\lambda\bar{\lambda}$ and $\mu\bar{\mu}$ are relatively prime. Since $\varphi(\Delta(K))$ divides $\Delta(L)$ we immediately obtain $\mu\bar{\mu} = 1$. Thus, $\Delta(L) = \varphi(\Delta(K))\lambda\bar{\lambda}$.

Let us prove (4.1.1) and (4.1.2). We may assume that $X \subset Y$ and that $Y \backslash X$ is the interior of the regular neighborhood $U \subset Y$ of the n-th component of K in Y. Let $p: \tilde{X} \to X$ and $q: \tilde{Y} \to Y$ be the maximal abelian coverings with the groups of covering transformations respectively $H_1(X) \approx \mathbf{Z}^n$ (generators $t_1, \ldots, t_n$) and $H_1(Y) \approx \mathbf{Z}^{n-1}$ (generators $t_1, \ldots, t_{n-1}$). It is clear that p is the composition of an infinite cyclic covering $\tilde{X} \to q^{-1}(X)$ and the covering $q: q^{-1}(X) \to X$.

Fix a C^1-triangulation of Y so that X and U are simplicial subcomplexes of Y. Fix also the induced equivariant triangulations in $\tilde{X}$ and $\tilde{Y}$.

The ring Λ_{n-1} determines via the natural homomorphism $\mathbf{Z}[\pi_1(Y)] \to \mathbf{Z}[H_1 Y] = \Lambda_{n-1}$ a system of local coefficients on Y which we denote by the same symbol Λ_{n-1}. According to definitions, for any simplicial subsets $A \supset B$ of Y the Λ_{n-1}-module $H_*(A, B; \Lambda_{n-1})$ equals $H_*(C(q^{-1}(A), q^{-1}(B); \mathbf{Z}))$. Here the simplicial chain complex $C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$ is a finitely generated free Λ_{n-1}-complex. Analogously Λ_n defines a system of local coefficients on X and for simplicial subsets $A \supset B$ of X the Λ_n-module $H_*(A, B; \Lambda_n)$ equals $H_*(C(p^{-1}(A), p^{-1}(B); \mathbf{Z}))$. Note that

$$\Lambda_{n-1} \otimes_{\Lambda_n} C_*(p^{-1}(A), p^{-1}(B); \mathbf{Z}) = C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$$

where Λ_n acts on Λ_{n-1} via φ.

Claim 1. For $i \neq 1, m-1$,

$$\operatorname{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \operatorname{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = \operatorname{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = 0 .$$

For $i = 1, m-1$,

$$\operatorname{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \operatorname{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = n-1; \ \operatorname{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = n-2 .$$

Proof of Claim 1. We shall compute the rank of $H_i(X; \Lambda_n)$; modules $H_i(X; \Lambda_{n-1})$ and $H_i(Y; \Lambda_{n-1})$ can be treated similarly.

Denote by V a wedge of n circles in X such that the inclusion homomorphism $H_1(V; \mathbf{Z}) \to H_1(X; \mathbf{Z}) = \mathbf{Z}^n$ is bijective. Then $H_i(X, V, \mathbf{Z}) = 0$ for $i \leqslant m-2$. Therefore an application of Lemma 3.2(i) to complexes $C_*(\tilde{X}, p^{-1}(V); \mathbf{Z})$ and $C_*(X, V; \mathbf{Z})$ gives that $\operatorname{rk}_{\Lambda_n} H_i(X, V; \Lambda_n) = 0$ for $i \leqslant m-2$. This implies that $\operatorname{rk} H_i(X; \Lambda_n) = \operatorname{rk} H_i(V; \Lambda_n)$ for $i \leqslant m-3$ and that $\operatorname{rk} H_{m-2}(X; \Lambda_n) \leqslant \operatorname{rk} H_{m-2}(V; \Lambda_n)$. The rank of $H_i(V; \Lambda_n)$ can be computed directly: It is equal to 0 if $i \neq 1$ and to $n-1$ if $i = 1$. Thus the rank of $H_i(X; \Lambda_n)$ equals 0 if $i \neq 1$, $m-1$ and equals $n-1$ if $i = 1$. The equality $\operatorname{rk} H_{m-1}(X; \Lambda_n) = n-1$ follows from duality or from the equalities

$$\sum_{i=0}^{m} (-1)^i \operatorname{rk} H_i(X; \Lambda_n) = \chi(X) = 0 .$$

Claim 2. The exact homology sequence of (Y, X) with coefficients in Λ_{n-1} splits into short exact sequences

$$\begin{array}{llll} 0 \to H_m(Y, X; \Lambda_{n-1}) & \to H_{m-1}(X; \Lambda_{n-1}) & \to H_{m-1}(Y; \Lambda_{n-1}) & \to 0, \\ 0 \to H_i(X; \Lambda_{n-1}) & \overset{\approx}{\to} H_i(Y; \Lambda_{n-1}) & \to 0, \quad (i \neq 1, m-1) & \\ 0 \to H_2(Y, X; \Lambda_{n-1}) & \overset{\partial_1}{\to} H_1(X; \Lambda_{n-1}) & \to H_1(Y; \Lambda_{n-1}) & \to 0. \end{array}$$

Proof of Claim 2. Clearly, $H_i(Y, X; \Lambda_{n-1}) = H_i(U, \partial U; \Lambda_{n-1}) = 0$ for $i \neq 2, m$. Therefore the only thing to prove is the injectivity of ∂_1. According to Claim 1 rk $H_1(X; \Lambda_{n-1}) = n - 1$ and rk $H_1(Y; \Lambda_{n-1}) = n - 2$. Since $H_2(Y, X; \Lambda_{n-1}) = \Lambda_{n-1}$ we see that ∂_1 is injective.

Proof of (4.1.1). In view of the equalities rg $H_i(X; \Lambda_n) =$ rg $H_i(X; \Lambda_{n-1})$, $i = 0, 1, \ldots$ we may apply Lemma 3.2 (iii) to the chain complexes $C_*(\tilde{X}; \mathbf{Z})$ and $C_*(q^{-1}(X); \mathbf{Z})$ respectively over Λ_n and Λ_{n-1}. Since $m - 1 > r > 1$ Claims 1, 2 show that $H_r(X; \Lambda_n)$ and $H_r(X; \Lambda_{n-1})$ are torsion modules respectively over Λ_n and Λ_{n-1} and $H_r(X, \Lambda_{n-1}) = H_r(Y; \Lambda_{n-1})$. By definition $\Delta(K) =$ ord $H_r(X; \Lambda_n)$ and $\Delta(L) =$ ord $H_r(Y; \Lambda_{n-1}) =$ ord $H_r(X; \Lambda_{n-1})$. Lemma 3.2 (iii) directly implies that $\varphi(\Delta(K))$ divides $\Delta(L)$.

It remains to prove Statement (4.1.2) which is, of course, the core of Theorem 1. For simplicial subsets $A \supset B$ of Y we shall denote by $C(A, B)$ the (simplicial) chain Q_{n-1}-complex $Q_{n-1} \otimes_{\Lambda_{n-1}} C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$. Clearly

$$H_i(A, B; Q_{n-1}) = H_i(C(A, B)) = Q_{n-1} \otimes_{\Lambda_{n-1}} H_i(A, B; \Lambda_{n-1}).$$

Consider the short exact sequence of chain Q_{n-1}-complexes

(5) $$0 \to C(X) \to C(Y) \to C(Y, X) \to 0.$$

Provide the homology modules of complexes $C(X)$, $C(Y)$, $C(Y, X)$ with bases as follows. It is evident that $H_i(C(Y, X)) = 0$ for $i \neq 2, m$ and

$$H_i(C(Y, X)) = H_i(C(U, \partial U)) = H_i(U, \partial U; Q_{n-1}) = Q_{n-1}$$

for $i = 2, m$. Fix a lifting $\tilde{U} \subset \tilde{Y}$ of $U \approx S^{m-2} \times D^2$. Fix in $H_m(C(Y, X))$ the generator $[\tilde{U}, \partial\tilde{U}]$. Fix in $H_2(C(Y, X))$ the generator $[\Delta, \partial\Delta]$ where Δ is the meridional disk of $\tilde{U}$.

It follows from Claim 1 that $H_i(C(X)) = H_i(C(Y)) = 0$ for $i \neq 1, m - 1$. Fix an arbitrary basis f in the $(n-2)$-dimensional vector Q_{n-1}-space $H_1(Y; Q_{n-1})$. Fix the dual basis g in $H_{m-1}(Y; Q_{n-1})$. It follows from Claim 2 that inclusion homomorphisms $H_i(C(X)) \to H_i(C(Y))$ are surjective for all i. Let F and G be sequences of $n - 2$ vectors in $H_1(C(X))$ and in $H_{m-1}(C(X))$ whose images under these inclusion homomorphisms are equal respectively to f and g. Claim 2 implies that $[\partial\tilde{U}]$, G is a basis in $H_{m-1}(C(X))$ and

$[\partial\Delta]$, F is a basis in $H_1(C(X))$. Now all homology modules of complexes $C(X)$, $C(Y)$, $C(Y, X)$ are provided with bases.

Provide the modules of $C(X)$, $C(Y)$, $C(Y, X)$ with natural bases (see Sec. 2.2). We may choose these bases to be compatible in the sense of Lemma 2.1.1. According to this Lemma

$$\tau(C(Y)) = \pm\, \tau(C(X))\tau(C(Y, X))\tau(\mathscr{H})$$

where $\mathscr{H}$ is the homology sequence associated with the exact sequence (5). It is evident that $\tau(\mathscr{H}) = \pm 1$. It is easy to verify that $\tau(C(Y, X)) = \tau(C(U, \partial U)) = \pm 1$. (Indeed, the pair $(U, \partial U)$ has a cell structure such that Int U contains 2 open cells; the meridional disc and its complement; for such cell structure the equality $\tau(C(U, \partial U)) = \pm 1$ is evident. The case of an arbitrary cell structure (or triangulation) follows from the invariance of torsion under cell subdivision). Thus $\tau(C(Y)) = \pm\, \tau(C(X))$. Note that $\tau(C(Y))$ represents $\omega(Y)$. Therefore $\tau(C(X))$ also represents $\omega(Y)$.

Consider the chain complex

$$C = Q_n^0 \otimes_{\Lambda_n} C_*(\tilde{X}; \mathbf{Z}).$$

Note that Q_n^0 is a local ring with the maximal ideal $(t_n - 1)Q_n^0$ and associated field Q_{n-1}. Clearly, $Q_{n-1} \otimes_{Q_n^0} C = C(X)$. The natural bases in chain modules of $C(X)$ lift to natural bases in chain modules of C. Claim 1 implies that for all $i \geqslant 0$

$$\mathrm{rk}_{Q_n^0} H_i(C) = \mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{Q_{n-1}} H_i(C(X)).$$

Therefore we may apply Lemma 3.4 to complexes C, $C(X)$. This lemma shows that: $H_i(C) = H_i(C(X)) = 0$ for $i \neq 1, m-1$; the basis $[\partial\Delta]$, F in $H_1(C(X))$ lifts to a basis, say, $f_0, f_1, \dots, f_{n-2}$ in $H_1(C)$; the basis $[\partial\tilde{U}]$, G in $H_{m-1}(C(X))$ lifts to a basis, say, $g_0, g_1, \dots, g_{n-2}$ in $H_{m-1}(C)$; the submodules of cycles and boundaries of C are free in all dimensions. Thus we may apply the constructions of Sec. 2.1 to C which gives rise to a torsion $\tau(C) \in Q_n^0$. It follows directly from the formula (3) that $\varphi(\tau(C)) = \tau(C(X))$. Thus $\varphi(\tau(C))$ represents $\omega(Y)$.

Let v be the matrix of the semi-linear intersection pairing

$$< \,,\, > : H_1(X; Q_n^0) \times H_{m-1}(X; Q_n^0) \to Q_n^0$$

with respect to bases $f_0, f_1, \dots, f_{n-2}$ and $g_0, g_1, \dots, g_{n-2}$. (Here $H_i(X; Q_n^0) = H_i(C)$). It is clear that $\tau(C)\,(\det v)^{-1}$ represents $\omega(X)$. Put $\omega = \tau(C)\,(\det v)^{-1}$. We shall prove that

$$\det v = \pm\,(t_n - 1) + (t_n - 1)^2 a \tag{6}$$

where $a \in Q_n^0$. Then $(t_n-1)\omega \in Q_n^0$ and

$$\varphi((t_n-1)\omega) = \varphi(\tau(C)[\pm 1+(t_n-1)a]^{-1}) = \pm\, \varphi(\tau(C)) \in \omega(Y).$$

This would complete the proof of (4.1.2).

It is obvious that

$$v = \begin{bmatrix} <f_0, g_0> & (t_n-1)\alpha \\ (t_n-1)\beta & E+(t_n-1)\gamma \end{bmatrix}$$

where α, β, γ are respectively a $(n-2)$-row, $(n-2)$-column and $(n-2) \times (n-2)$-matrix over Q_n^0. It turns out that

$$(7) \qquad <f_0, g_0> = \pm\,(t_n-1) + (t_n-1)^2 b$$

with $b \in Q_n^0$. This immediately implies (6).

I shall prove (7) for a special choice of f_0 which is sufficient for our aims. Let $\theta\colon [0, 1] \to \partial\tilde{X}$ be a path whose projection to $\tilde{Y}$ is a loop parametrizing $\partial\Delta \subset \partial\tilde{U}$. Let $\eta\colon [0, 1] \to \tilde{X}$ be a path such that $\eta(0) = \theta(0)$ and $\eta(1) = t_1 \cdot \theta(0)$. Consider the singular chain $\vartheta = \theta - t_1\theta + t_n\eta - \eta$. It is easy to check up that ϑ is a cycle in $\tilde{X}$ and that its homology class $[\vartheta] \in H_1(C)$ projects to $(1-t_1)\,[\partial\Delta] \in H_1(C(X))$. Put $f_0 = (1-t_1)^{-1}[\vartheta]$. Then $<f_0, g_0> = (1-t_1)^{-1}<[\vartheta], g_0> = (1-t_1)^{-1}(t_n-1) <\eta, g_0>$ where in the right part the brackets $< \ , \ >$ denote the intersection pairing

$$H_1(X, \partial X; Q_n^0) \times H_{m-1}(X; Q_n^0) \to Q_n^0.$$

The image of $<\eta, g_0>$ under $\varphi\colon Q_n^0 \to Q_{n-1}$ can be computed using the analogous pairing

$$H_1(X, \partial X; Q_{n-1}) \times H_{m-1}(X; Q_{n-1}) \to Q_{n-1}.$$

Namely, $\varphi(<\eta, g_0>) = \pm\,(t_1-1)$. Thus $<\eta, g_0> = \pm\,(t_1-1) + (t_n-1)c$ with $c \in Q_n^0$. Therefore $<f_0, g_0> = \pm\,(t_n-1) + (t_n-1)^2 b$ where $b = (1-t_1)^{-1}c$. This implies (7).

4.2. *Proof of Theorem 2.* We may assume that $\Delta_{u-1}(L) \neq 0$ and $l_1 = l_2 = \cdots = l_{n-1} = 0$. Then the n-th component of K lifts to the maximal abelian covering of the exterior Y of L. The remaining part of the proof is analogous to the proof of Theorem 1. Note, however, the necessary changes. In Claim 1 for $i = 1, 2$

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Delta_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = u - 1; \ \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = u - 2.$$

In the proof of (4.1.1) one should take into account that $\mathrm{Tors}_{\Lambda_{n-1}} H_1(X; \Lambda_{n-1})$ injects into $\mathrm{Tors}_{\Lambda_{n-1}} H_1(Y; \Lambda_{n-1})$ and thus the order of the first of these 2 modules divides the order of the second one.

REFERENCES

[1] HILLMAN, J. A. *Alexander Ideals of Links.* Lecture Notes in Math. 895, Springer-Verlag, New York, 1981.

[2] LEVINE, J. P. Links with Alexander Polynomial zero. *Indiana Univ. Math. J. 36* (1987), 91-108.

[3] MILNOR, J. W. Whitehead Torsion. *Bull. Amer. Math. Soc. 72* (1966), 358-426.

[4] TURAEV, V. G. Reidemeister Torsion in the Knot Theory. *Uspechi Matem. Nauk 41* (1986), 97-147 (Russian); English translation: *Russian Math. Surveys 41* (1986), 119-182.

[5] TORRES, G. On the Alexander polynomial. *Annals of Math. 57* (1953), 57-89.

(Reçu le 1er juillet 1987)

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