

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 34 (1988)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON TORRES-TYPE RELATIONS FOR THE ALEXANDER
POLYNOMIALS OF LINKS
Autor: Turaev, V. G.
Kapitel: §3. Algebraic lemmas
DOI: <https://doi.org/10.5169/seals-56589>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 12.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

where $r = (m-1)/2$ and $\varepsilon(i) = (-1)^{i+1}$. It is easy to show that under a different choice of natural bases and bases h_0, h_1, \dots, h_m the element d is replaced by $\pm gq\bar{q}d$ with $g \in G, q \in Q \setminus 0$. Thus the set $\{\pm gq\bar{q}d \mid g \in Q \setminus 0\} \subset Q$ does not depend on the choice of bases. It also does not depend on the choice of triangulation in M . It is this set which is $\omega(M)$.

An explicit formula established in [4] enables us to calculate $\omega(M)$ in terms of the orders of $\mathbf{Z}[G]$ -modules $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z}), H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$ and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by J the image of the inclusion homomorphism $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$ where $r = (m-1)/2$. Then up to multiples of type $q\bar{q}$ with $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord}(\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$ imply that $H_*(\partial\tilde{M})$ and J are torsion $\mathbf{Z}[G]$ -modules. Therefore $\text{ord } H_i(\partial\tilde{M})$ and $\text{ord } J$ are non-zero elements of $\mathbf{Z}[G]$.

We shall apply formula (4) in the case where M is the exterior of an n -component link $K \subset S^m$ with odd m . The condition $H_*(\partial M; Q) = 0$ is always fulfilled in this case. Here the field Q is canonically identified with the field of rational functions of n variables $Q_n = Q(t_1, \dots, t_n)$. Thus $\omega(M) \subset Q_n$. If $m \geq 5$ then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If $m = 3$ then there exists a unique subset $\alpha = \alpha(K)$ of the set $\{1, 2, \dots, n\}$ such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

§ 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module H over a (commutative) domain R we denote by $\text{rk}_R H$ or, briefly, by $\text{rk } H$ the integer $\dim_Q(Q \otimes_R H)$ where $Q = Q(R)$ denotes the field of fractions of R . For a R -linear homomorphism $f: H \rightarrow H'$ we put $\text{rk } f = \text{rk}_R f(H)$. Note that if \bar{R} is the localization of R at some multiplicative system then $Q(\bar{R}) = Q(R)$ and therefore the (exact) functor $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$

preserves the ranks of modules and homomorphisms. If H, H' are finitely generated free R -modules and if A is the matrix of a R -homomorphism $H \rightarrow H'$ with respect to some bases then $\text{rk } f = \text{rk } A$ where $\text{rk } A$ is the maximal integer r such that some $r \times r$ -minor of A is non-zero.

If R is a unique factorization domain with 1 and if A is a matrix with $n < \infty$ columns and possibly infinite number of rows then $\Delta_i(A)$ denotes the greatest common divisor of the $(n-i+1) \times (n-i+1)$ -minors of A . Here $i = 1, 2, \dots$ and $\Delta_i(A)$ is an element of R defined up to a unit multiple. If H is a finitely generated module over R and A is a presentation matrix of H then $\Delta_i(A)$ depends only on H and i ; one defines $\Delta_i(H) = \Delta_i(A)$. Clearly $\Delta_i(H) = 0$ for $i \leq \text{rg } H = n - \text{rg } A$ and $\Delta_i(H) \neq 0$ for $i > \text{rg } H$. The invariant $\Delta_1(H)$ is denoted also by $\text{ord } H$; it is called the order of H . It is clear that $\text{ord } H \neq 0$ iff $H = \text{Tors}_R H$. For proofs and further information see [1].

Recall, finally, that a local ring is a domain K which has a unique maximal (proper) ideal. The quotient of K by this ideal is a field which we shall call "the field associated to K ".

3.2. LEMMA. *Let R, R' be (commutative) domains with 1 and let $\varphi: R \rightarrow R'$ be a ring homomorphism. Let $C = (\dots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \dots)$ be a finitely generated free chain complex over R and let C' be the chain R' -complex $R' \otimes_R C$. Then: (i) $\text{rk}_{R'} H_i(C') \geq \text{rk}_R H_i(C)$ and $\text{rk } \partial'_i \leq \text{rk } \partial_i$ for all i where ∂_i, ∂'_i are the boundary homomorphisms $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$; (ii) if $\text{rk } H_i(C') = \text{rk } H_i(C)$ for some i then $\text{rk } \partial'_j = \text{rk } \partial_j$ for $j = i, i+1$; (iii) if R, R' are unique factorization Noetherian domains and if $\text{rk } H_i(C') = \text{rk } H_i(C)$ then $\varphi(\text{ord}(\text{Tors}_R H_i(C)))$ divides $\text{ord}(\text{Tors}_{R'} H_i(C'))$.*

Proof. Let $n = \text{rk } C_i$. Let $A = (a_{p,q})$, $1 \leq q \leq n$, $1 \leq p$, be the matrix of ∂_i with respect to some bases in C_i, C_{i+1} . Then $A' = (\varphi(a_{p,q}))$ is the matrix of ∂'_i with respect to the induced bases in C'_i, C'_{i+1} . It is evident that $\text{rk } \partial'_i = \text{rk } A' \leq \text{rk } A = \text{rk } \partial_i$. Therefore

$$\text{rk } H_i(C') = n - \text{rk } \partial'_i - \text{rk } \partial'_{i+1} \geq n - \text{rk } \partial_i - \text{rk } \partial_{i+1} = \text{rk } H_i(C).$$

These inequalities imply (i) and (ii).

Put $r = n - \text{rk } A + 1$ and denote the R -module $C_i/\text{Im } \partial_i$ by J . Since A is a presentation matrix of J we have $\text{ord}(\text{Tors}_R J) = \Delta_r(A)$ (see [1, p. 31]). From the exact sequence $0 \rightarrow H_i(C) \rightarrow J \rightarrow C_{i-1}$ we obtain that $\text{Tors } J = \text{Tors } H_i(C)$. Thus $\text{ord}(\text{Tors } H_i(C)) = \Delta_r(A)$. Analogously $\text{ord}(\text{Tors } H_i(C')) = \Delta_{r'}(A')$ where $r' = n - \text{rk } A' + 1$. If $\text{rk } H_i(C) = \text{rk } H_i(C')$ then $\text{rk } A = \text{rk } A'$ and therefore $r = r'$. It is evident that $\varphi(\Delta_j(A))$ divides $\Delta_j(A')$ for all j . This implies (iii).

3.3. LEMMA. Let R be a local ring and F be the associated field. Let $f: C_1 \rightarrow C_0$ be a R -homomorphism of finitely generated free R -modules and let $\bar{f}: F \otimes_R C_1 \rightarrow F \otimes_R C_0$ be the induced F -homomorphism. If $\text{rk } f = \text{rk } \bar{f}$ then with respect to some bases in C_1, C_0 the homomorphism f is presented by the matrix $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where E is the unit matrix of order $\text{rk } f$.

Proof. Since F is a field we can choose bases d_0, d_1 respectively in $F \otimes_R C_0, F \otimes_R C_1$ so that the matrix of \bar{f} regarding these bases has the form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Let \mathcal{D}_i be a lifting of d_i to $C_i, i = 1, 2$. Here \mathcal{D}_i is a sequence of $\text{rg } C_i$ elements of C_i . In view of Nakayama's lemma \mathcal{D}_i generate C_i . This implies that \mathcal{D}_i generates the $(\text{rg } C_i)$ -dimensional vector space $Q(R) \otimes_R C_i$ over the field $Q(R)$. Therefore, the elements of the sequence \mathcal{D}_i are linearly independent over $Q(R)$ and, hence, over R . Thus \mathcal{D}_i is a basis of C_i for $i = 0, 1$. The matrix of f with respect to bases $\mathcal{D}_0, \mathcal{D}_1$ has the form $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$ where U, X, Y, Z are matrices over the maximal ideal u of R . Note that $\det(E+U) = 1 \pmod{u}$. Since all elements of $R \setminus u$ are invertible in R the square matrix $E+U$ is invertible over R . Therefore we can choose bases in C_0, C_1 so that the corresponding matrix of f equals $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$. Since $\text{rk } f = \text{rk } \bar{f} = \text{rk } E, Y' = 0$.

3.4. LEMMA. Let R be a local ring and F be the associated field. Let $C = (\dots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \dots)$ be a finitely generated free chain complex over R . Let C' be the chain F -complex $F \otimes_R C$. Let ∂_i, ∂'_i be the boundary homomorphisms $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$. If $\text{rk}_R H_i(C) = \text{rk}_F H_i(C')$ for some i then: $H_i(C), \text{Im } \partial_{i+1}, \text{Im } \partial_i$ are free R -modules and $C_i = \text{Im } \partial_{i+1} \oplus H_i(C) \oplus \text{Im } \partial_i$; the projection $C \rightarrow C'$ induces F -isomorphisms $F \otimes_R H_i(C) \rightarrow H_i(C'), F \otimes_R \text{Im } \partial_j \rightarrow \text{Im } \partial'_j$ with $j = i, i+1$.

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

§ 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by Q_n the fraction field of the ring $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. Denote by Q_n^0 the subring of Q_n which consists of rational functions fg^{-1} with $f, g \in \Lambda_n$ and $g \notin (t_n - 1)\Lambda_n$ (so that