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Autor: Turaev, V. G.

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where $r = (m-1)/2$ and $\varepsilon(i) = (-1)^{i+1}$. It is easy to show that under a different choice of natural bases and bases $h_0, h_1, ..., h_m$ the element d is replaced by $\pm g q \bar{q} d$ with $g \in G$, $q \in Q \backslash 0$. Thus the set $\{\pm g q \bar{q} d \mid g \in Q \backslash 0\} \subset Q$ does not depend on the choice of bases. It also does not depend on the choice of triangulation in M. It is this set which is $\omega(M)$.

An explicit formula established in [4] enables us to calculate $\omega(M)$ in terms of the orders of $\mathbf{Z}[G]$-modules $H_*(\partial \tilde{M}) = H_*(\partial \tilde{M}; \mathbf{Z})$, $H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$ and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by J the image of the inclusion homomorphism $H_r(\partial \tilde{M}) \to H_r(\tilde{M})$ where $r = (m-1)/2$. Then up to multiples of type $q\bar{q}$ with $q \in Q \backslash 0$

$$\omega(M) = \operatorname{ord}\left(\operatorname{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)\right) (\operatorname{ord} J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\operatorname{ord} H_i(\partial M)]^{\varepsilon(i)} \tag{4}$$

(see [4, Theorem 5.1.1]). Note that the equalities $Q \otimes_{\mathbf{Z}[G]} H_*(\partial \tilde{M}) = H_*(\partial \tilde{M}; Q) = 0$ imply that $H_*(\partial \tilde{M})$ and J are torsion $\mathbf{Z}[G]$-modules. Therefore $\operatorname{ord} H_i(\partial \tilde{M})$ and $\operatorname{ord} J$ are non-zero elements of $\mathbf{Z}[G]$.

We shall apply formula (4) in the case where M is the exterior of an n-component link $K \subset S^m$ with odd m. The condition $H_*(\partial M; Q) = 0$ is always fulfilled in this case. Here the field Q is canonically identified with the field of rational functions of n variables $Q_n = Q(t_1, ..., t_n)$. Thus $\omega(M) \subset Q_n$. If $m \geqslant 5$ then (4) implies that

$$\Delta(K)(t_1, ..., t_n) \cdot \prod_{i=1}^{n} (t_i - 1) \subset \omega(M).$$

If $m = 3$ then there exists a unique subset $\alpha = \alpha(K)$ of the set $\{1, 2, ..., n\}$ such that

$$\Delta_{u(K)}(K)(t_1, ..., t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

§ 3. Algebraic lemmas

3.1. Preliminary definitions. For a finitely generated module H over a (commutative) domain R we denote by $\operatorname{rk}_R H$ or, briefly, by $\operatorname{rk} H$ the integer $\dim_Q(Q \otimes_R H)$ where $Q = Q(R)$ denotes the field of fractions of R. For a R-linear homomorphism $f: H \to H'$ we put $\operatorname{rk} f = \operatorname{rk}_R f(H)$. Note that if $\bar{R}$ is the localization of R at some multiplicative system then $Q(\bar{R}) = Q(R)$ and therefore the (exact) functor $(H \mapsto \bar{R} \otimes_R H, f \mapsto \operatorname{id}_{\bar{R}} \otimes f)$

preserves the ranks of modules and homomorphisms. If H, H' are finitely generated free R-modules and if A is the matrix of a R-homomorphism $H \to H'$ with respect to some bases then $\operatorname{rk} f = \operatorname{rk} A$ where $\operatorname{rk} A$ is the maximal integer r such that some $r \times r$-minor of A is non-zero.

If R is a unique factorization domain with 1 and if A is a matrix with $n < \infty$ columns and possibly infinite number of rows then $\Delta_i(A)$ denotes the greatest common divisor of the $(n-i+1) \times (n-i+1)$-minors of A. Here $i = 1, 2, \ldots$ and $\Delta_i(A)$ is an element of R defined up to a unit multiple. If H is a finitely generated module over R and A is a presentation matrix of H then $\Delta_i(A)$ depends only on H and i; one defines $\Delta_i(H) = \Delta_i(A)$. Clearly $\Delta_i(H) = 0$ for $i \leqslant \operatorname{rg} H = n - \operatorname{rg} A$ and $\Delta_i(H) \neq 0$ for $i > \operatorname{rg} H$. The invariant $\Delta_1(H)$ is denoted also by $\operatorname{ord} H$; it is called the order of H. It is clear that $\operatorname{ord} H \neq 0$ iff $H = \operatorname{Tors}_R H$. For proofs and further information see [1].

Recall, finally, that a local ring is a domain K which has a unique maximal (proper) ideal. The quotient of K by this ideal is a field which we shall call "the field associated to K".

3.2. LEMMA. *Let R, R' be (commutative) domains with 1 and let $\varphi: R \to R'$ be a ring homomorphism. Let $C = (\cdots \to C_{i+1} \to C_i \to \cdots)$ be a finitely generated free chain complex over R and let C' be the chain R'-complex $R' \otimes_R C$. Then:* (i) $\operatorname{rk}_{R'} H_i(C') \geqslant \operatorname{rk}_R H_i(C)$ *and* $\operatorname{rk} \partial'_i \leqslant \operatorname{rk} \partial_i$ *for all i where ∂_i, ∂'_i are the boundary homomorphisms $C_{i+1} \to C_i$, $C'_{i+1} \to C'_i$;* (ii) *if* $\operatorname{rk} H_i(C') = \operatorname{rk} H_i(C)$ *for some i then* $\operatorname{rk} \partial'_j = \operatorname{rk} \partial_j$ *for $j = i, i+1$;* (iii) *if R, R' are unique factorization Noetherian domains and if* $\operatorname{rk} H_i(C') = \operatorname{rk} H_i(C)$ *then* $\varphi(\operatorname{ord}(\operatorname{Tors}_R H_i(C)))$ *divides* $\operatorname{ord}(\operatorname{Tors}_{R'} H_i(C'))$.

Proof. Let $n = \operatorname{rk} C_i$. Let $A = (a_{p,q})$, $1 \leqslant q \leqslant n$, $1 \leqslant p$, be the matrix of ∂_i with respect to some bases in C_i, C_{i+1}. Then $A' = (\varphi(a_{p,q}))$ is the matrix of ∂'_i with respect to the induced bases in C'_i, C'_{i+1}. It is evident that $\operatorname{rk} \partial'_i = \operatorname{rk} A' \leqslant \operatorname{rk} A = \operatorname{rk} \partial_i$. Therefore

$$\operatorname{rk} H_i(C') = n - \operatorname{rk} \partial'_i - \operatorname{rk} \partial'_{i+1} \geqslant n - \operatorname{rk} \partial_i - \operatorname{rk} \partial_{i+1} = \operatorname{rk} H_i(C).$$

These inequalities imply (i) and (ii).

Put $r = n - \operatorname{rk} A + 1$ and denote the R-module $C_i/\operatorname{Im}\partial_i$ by J. Since A is a presentation matrix of J we have $\operatorname{ord}(\operatorname{Tors}_R J) = \Delta_r(A)$ (see [1, p. 31]). From the exact sequence $0 \to H_i(C) \to J \to C_{i-1}$ we obtain that $\operatorname{Tors} J = \operatorname{Tors} H_i(C)$. Thus $\operatorname{ord}(\operatorname{Tors} H_i(C)) = \Delta_r(A)$. Analogously $\operatorname{ord}(\operatorname{Tors} H_i(C')) = \Delta_{r'}(A')$ where $r' = n - \operatorname{rk} A' + 1$. If $\operatorname{rk} H_i(C) = \operatorname{rk} H_i(C')$ then $\operatorname{rk} A = \operatorname{rk} A'$ and therefore $r = r'$. It is evident that $\varphi(\Delta_j(A))$ divides $\Delta_j(A')$ for all j. This implies (iii).

3.3. LEMMA. *Let* R *be a local ring and* F *be the associated field. Let* $f: C_1 \to C_0$ *be a R-homomorphism of finitely generated free R-modules and let* $\bar{f}: F \otimes_R C_1 \to F \otimes_R C_0$ *be the induced F-homomorphism. If* $\operatorname{rk} f = \operatorname{rk} \bar{f}$ *then with respect to some bases in* C_1, C_0 *the homomorphism* f *is presented by the matrix* $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ *where* E *is the unit matrix of order* $\operatorname{rk} f$.

Proof. Since F is a field we can choose bases d_0, d_1 respectively in $F \otimes_k C_0$, $F \otimes_K C_1$ so that the matrix of $\bar{f}$ regarding these bases has the form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Let $\mathscr{D}_i$ be a lifting of d_i to $C_i, i = 1, 2$. Here $\mathscr{D}_i$ is a sequence of $\operatorname{rg} C_i$ elements of C_i. In view of Nakayama's lemma $\mathscr{D}_i$ generate C_i. This implies that $\mathscr{D}_i$ generates the $(\operatorname{rg} C_i)$-dimensional vector space $Q(R) \otimes_R C_i$ over the field $Q(R)$. Therefore, the elements of the sequence $\mathscr{D}_i$ are linearly independent over $Q(R)$ and, hence, over R. Thus $\mathscr{D}_i$ is a basis of C_i for $i = 0, 1$. The matrix of f with respect to bases $\mathscr{D}_0, \mathscr{D}_1$ has the form $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$ where U, X, Y, Z are matrices over the maximal ideal u of R. Note that $\det(E+U) = 1 \pmod{u}$. Since all elements of $R \backslash U$ are invertible in R the square matrix $E + U$ is invertible over R. Therefore we can choose bases in C_0, C_1 so that the corresponding matrix of f equals $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$. Since $\operatorname{rk} f = \operatorname{rk} \bar{f} = \operatorname{rk} E$, $Y' = 0$.

3.4. LEMMA. *Let* R *be a local ring and* F *be the associated field. Let* $C = (\cdots \to C_{i+1} \to C_i \to \cdots)$ *be a finitely generated free chain complex over* R. *Let* C' *be the chain F-complex* $F \otimes_R C$. *Let* ∂_i, ∂'_i *be the boundary homomorphisms* $C_{i+1} \to C_i, C'_{i+1} \to C'_i$. *If* $\operatorname{rk}_R H_i(C) = \operatorname{rk}_F H_i(C')$ *for some* i *then:* $H_i(C), \operatorname{Im} \partial_{i+1}, \operatorname{Im} \partial_i$ *are free R-modules and* $C_i = \operatorname{Im} \partial_{i+1} \oplus H_i(C) \oplus \operatorname{Im} \partial_i$; *the projection* $C \to C'$ *induces F-isomorphisms* $F \otimes_R H_i(C) \to H_i(C'), F \otimes_R \operatorname{Im} \partial_j \to \operatorname{Im} \partial'_j$ *with* $j = i, i+1$.

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

§ 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by Q_n the fraction field of the ring $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. Denote by Q_n^0 the subring of Q_n which consists of rational functions fg^{-1} with $f, g \in \Lambda_n$ and $g \notin (t_n - 1)\Lambda_n$ (so that