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POLYNOMIALS OF LINKS
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The non-trivial case of Theorem 2 is the case $l_1 = l_2 = \dots = l_{n-1} = 0$: otherwise $u(L) \geq u$ so that $\Delta_{u-1}(L) = 0$ and we may put $\lambda = 0$.

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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§ 2. TORSIONS OF CHAIN COMPLEXES AND MANIFOLDS

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let Q be a field. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are two bases of a Q -module then $a_i = \sum_{j=1}^n c_{i,j} b_j$ where $(c_{i,j})$ is a non-singular $n \times n$ -matrix over Q ; the determinant $\det(c_{i,j}) \in Q \setminus 0$ is denoted by $[a/b]$.

Let $C = (C_m \rightarrow \dots \rightarrow C_0)$ be a chain Q -complex. Suppose that each Q -module C_i is finite dimensional with a preferred basis c_i and each Q -module $H_i(C)$ also has a preferred basis h_i . (The case $C_i = 0$ or $H_i(C) = 0$ is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion $\tau(C) \in Q$ as follows. For each $i = 1, 2, \dots, m$ choose a sequence $b_i = (b_1^i, \dots, b_{r_i}^i)$ of elements of C_i such that $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$ is a basis in $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$. For each $i = 0, 1, \dots, m$ choose a lifting \tilde{h}_i of the basis h_i to $\text{Ker } \partial_{i-1}$. The combined sequence $\partial_i(b_{i+1})\tilde{h}_i b_i$ is a basis in C_i . (It is understood that $b_0 = \emptyset$ and $b_{m+1} = \emptyset$). Put

$$(3) \quad \tau(C) = \prod_{i=0}^m [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where $\varepsilon(i) = (-1)^{i+1}$. Clearly, $\tau(C) \in Q \setminus 0$. It is easy to verify that $\tau(C)$ does not depend on the choice of b_i and \tilde{h}_i .

(Note that the torsion of C defined in Milnor's survey article [3] equals $\pm \tau(C)^{-1} \in Q/\pm 1$ and that Milnor uses the additive notation for the multiplication in $Q \setminus 0 = K_1(Q)$.)

2.1.1. LEMMA (multiplicativity of torsion). Let $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ be a short exact sequence of m -dimensional chain complexes over a field Q .

Suppose that for all $i = 0, 1, \dots, m$ the modules C_i, C'_i, C''_i are provided with preferred bases c'_i, c_i, c''_i which are compatible, in the sense that $[c'_i c''_i / c_i] = \pm 1$. Suppose that for all $i = 0, 1, \dots, m$ the homology modules $H_i(C), H_i(C'), H_i(C'')$ are provided with preferred bases. Let \mathcal{H} be the homology sequence of the sequence $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$:

$$\mathcal{H} = (H_m(C') \rightarrow H_m(C) \rightarrow \dots \rightarrow H_0(C) \rightarrow H_0(C'')).$$

Consider \mathcal{H} as an acyclic based chain complex over Q . Then $\tau(C) = \pm \tau(C')\tau(C'')\tau(\mathcal{H})$.

For a proof see [3].

2.2. THE TORSION ω . Let M be an orientable compact smooth manifold of odd dimension m with $\text{rg } H_1(M) \geq 1$. Denote the free abelian group $H_1(M)/\text{Tors } H_1(M)$ by G . Denote the fraction field of the group ring $\mathbb{Z}[G]$ by Q . Provide Q with the involution $q \mapsto \bar{q}$ which sends $g \in G$ to g^{-1} . The field Q defines via the natural homomorphism $\mathbb{Z}[\pi_1(M)] \rightarrow Q$ a system of local coefficients on M . We shall denote this system by the same symbol Q . Assume that $H_*(\partial M; Q) = 0$. In this setting one can consider a torsion-type invariant $\omega(M)$ of M which is "an element of $Q \setminus 0$ " defined up to multiplication by $\pm gq\bar{q}$ with $g \in G$ and $q \in Q \setminus 0$ (see [4]).

Recall the definition of $\omega(M)$ given in [4, § 5]. Let $\tilde{M} \rightarrow M$ be the regular covering of M corresponding to the kernel of the natural homomorphism $\pi_1(M) \rightarrow G$. Fix a C^1 -triangulation of M and the induced G -equivariant triangulation of \tilde{M} . Choose over each simplex of the (fixed) triangulation of M a simplex of the triangulation of \tilde{M} . These simplices in \tilde{M} being arbitrarily oriented and ordered determine "natural" bases of the modules of the simplicial chain $\mathbb{Z}[G]$ -complex $C_*(\tilde{M}; \mathbb{Z})$. These bases induce "natural" Q -bases in the chain Q -complex

$$C = Q \otimes_{\mathbb{Z}[G]} C_*(\tilde{M}; \mathbb{Z}).$$

For all $i = 0, 1, \dots, m$ choose an arbitrary Q -basis h_i in $H_i(M; Q) = H_i(C)$. Denote by $\tau(C, h_0, \dots, h_m)$ the torsion of C with respect to the bases in chain modules constructed above and the bases h_0, h_1, \dots, h_m in homology. Since $H_*(\partial M; Q) = 0$ the semi-linear intersection form $H_i(M; Q) \times H_{m-i}(M; Q) \rightarrow Q$ is non-singular. Let v_i be the matrix of this form regarding the bases h_i and h_{m-i} . Put

$$d = \tau(C, h_0, h_1, \dots, h_m) \prod_{i=0}^r (\det v_i)^{-\varepsilon(i)} \in Q \setminus 0$$

where $r = (m-1)/2$ and $\varepsilon(i) = (-1)^{i+1}$. It is easy to show that under a different choice of natural bases and bases h_0, h_1, \dots, h_m the element d is replaced by $\pm gq\bar{q}d$ with $g \in G, q \in Q \setminus 0$. Thus the set $\{\pm gq\bar{q}d \mid g \in G, q \in Q \setminus 0\} \subset Q$ does not depend on the choice of bases. It also does not depend on the choice of triangulation in M . It is this set which is $\omega(M)$.

An explicit formula established in [4] enables us to calculate $\omega(M)$ in terms of the orders of $\mathbf{Z}[G]$ -modules $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z})$, $H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$ and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by J the image of the inclusion homomorphism $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$ where $r = (m-1)/2$. Then up to multiples of type $q\bar{q}$ with $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord}(\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$ imply that $H_*(\partial\tilde{M})$ and J are torsion $\mathbf{Z}[G]$ -modules. Therefore $\text{ord } H_i(\partial\tilde{M})$ and $\text{ord } J$ are non-zero elements of $\mathbf{Z}[G]$.

We shall apply formula (4) in the case where M is the exterior of an n -component link $K \subset S^m$ with odd m . The condition $H_*(\partial M; Q) = 0$ is always fulfilled in this case. Here the field Q is canonically identified with the field of rational functions of n variables $Q_n = Q(t_1, \dots, t_n)$. Thus $\omega(M) \subset Q_n$. If $m \geq 5$ then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If $m = 3$ then there exists a unique subset $\alpha = \alpha(K)$ of the set $\{1, 2, \dots, n\}$ such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

§ 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module H over a (commutative) domain R we denote by $\text{rk}_R H$ or, briefly, by $\text{rk } H$ the integer $\dim_Q(Q \otimes_R H)$ where $Q = Q(R)$ denotes the field of fractions of R . For a R -linear homomorphism $f: H \rightarrow H'$ we put $\text{rk } f = \text{rk}_R f(H)$. Note that if \bar{R} is the localization of R at some multiplicative system then $Q(\bar{R}) = Q(R)$ and therefore the (exact) functor $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$