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**Autor:** Turaev, V. G.

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The non-trivial case of Theorem 2 is the case  $l_1 = l_2 = \dots = l_{n-1} = 0$ : otherwise  $u(L) \geq u$  so that  $\Delta_{u-1}(L) = 0$  and we may put  $\lambda = 0$ .

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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## § 2. TORSIONS OF CHAIN COMPLEXES AND MANIFOLDS

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let  $Q$  be a field. If  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  are two bases of a  $Q$ -module then  $a_i = \sum_{j=1}^n c_{i,j} b_j$  where  $(c_{i,j})$  is a non-singular  $n \times n$ -matrix over  $Q$ ; the determinant  $\det(c_{i,j}) \in Q \setminus 0$  is denoted by  $[a/b]$ .

Let  $C = (C_m \rightarrow \dots \rightarrow C_0)$  be a chain  $Q$ -complex. Suppose that each  $Q$ -module  $C_i$  is finite dimensional with a preferred basis  $c_i$  and each  $Q$ -module  $H_i(C)$  also has a preferred basis  $h_i$ . (The case  $C_i = 0$  or  $H_i(C) = 0$  is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion  $\tau(C) \in Q$  as follows. For each  $i = 1, 2, \dots, m$  choose a sequence  $b_i = (b_1^i, \dots, b_{r_i}^i)$  of elements of  $C_i$  such that  $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$  is a basis in  $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$ . For each  $i = 0, 1, \dots, m$  choose a lifting  $\tilde{h}_i$  of the basis  $h_i$  to  $\text{Ker } \partial_{i-1}$ . The combined sequence  $\partial_i(b_{i+1})\tilde{h}_i b_i$  is a basis in  $C_i$ . (It is understood that  $b_0 = \emptyset$  and  $b_{m+1} = \emptyset$ .) Put

$$(3) \quad \tau(C) = \prod_{i=0}^m [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where  $\varepsilon(i) = (-1)^{i+1}$ . Clearly,  $\tau(C) \in Q \setminus 0$ . It is easy to verify that  $\tau(C)$  does not depend on the choice of  $b_i$  and  $\tilde{h}_i$ .

(Note that the torsion of  $C$  defined in Milnor's survey article [3] equals  $\pm \tau(C)^{-1} \in Q/\pm 1$  and that Milnor uses the additive notation for the multiplication in  $Q \setminus 0 = K_1(Q)$ .)

2.1.1. LEMMA (multiplicativity of torsion). *Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  be a short exact sequence of  $m$ -dimensional chain complexes over a field  $Q$ .*

Suppose that for all  $i = 0, 1, \dots, m$  the modules  $C_i, C'_i, C''_i$  are provided with preferred bases  $c'_i, c_i, c''_i$  which are compatible, in the sense that  $[c'_i c''_i / c_i] = \pm 1$ . Suppose that for all  $i = 0, 1, \dots, m$  the homology modules  $H_i(C), H_i(C'), H_i(C'')$  are provided with preferred bases. Let  $\mathcal{H}$  be the homology sequence of the sequence  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ :

$$\mathcal{H} = (H_m(C') \rightarrow H_m(C) \rightarrow \dots \rightarrow H_0(C) \rightarrow H_0(C'')).$$

Consider  $\mathcal{H}$  as an acyclic based chain complex over  $Q$ . Then  $\tau(C) = \pm \tau(C')\tau(C'')\tau(\mathcal{H})$ .

For a proof see [3].

2.2. THE TORSION  $\omega$ . Let  $M$  be an orientable compact smooth manifold of odd dimension  $m$  with  $\text{rg } H_1(M) \geq 1$ . Denote the free abelian group  $H_1(M)/\text{Tors } H_1(M)$  by  $G$ . Denote the fraction field of the group ring  $\mathbf{Z}[G]$  by  $Q$ . Provide  $Q$  with the involution  $q \mapsto \bar{q}$  which sends  $g \in G$  to  $g^{-1}$ . The field  $Q$  defines via the natural homomorphism  $\mathbf{Z}[\pi_1(M)] \rightarrow Q$  a system of local coefficients on  $M$ . We shall denote this system by the same symbol  $Q$ . Assume that  $H_*(\partial M; Q) = 0$ . In this setting one can consider a torsion-type invariant  $\omega(M)$  of  $M$  which is “an element of  $Q \setminus 0$  defined up to multiplication by  $\pm gq\bar{q}$  with  $g \in G$  and  $q \in Q \setminus 0$ ” (see [4]).

Recall the definition of  $\omega(M)$  given in [4, § 5]. Let  $\tilde{M} \rightarrow M$  be the regular covering of  $M$  corresponding to the kernel of the natural homomorphism  $\pi_1(M) \rightarrow G$ . Fix a  $C^1$ -triangulation of  $M$  and the induced  $G$ -equivariant triangulation of  $\tilde{M}$ . Choose over each simplex of the (fixed) triangulation of  $M$  a simplex of the triangulation of  $\tilde{M}$ . These simplices in  $\tilde{M}$  being arbitrarily oriented and ordered determine “natural” bases of the modules of the simplicial chain  $\mathbf{Z}[G]$ -complex  $C_*(\tilde{M}; \mathbf{Z})$ . These bases induce “natural”  $Q$ -bases in the chain  $Q$ -complex

$$C = Q \otimes_{\mathbf{Z}[G]} C_*(\tilde{M}; \mathbf{Z}).$$

For all  $i = 0, 1, \dots, m$  choose an arbitrary  $Q$ -basis  $h_i$  in  $H_i(M; Q) = H_i(C)$ . Denote by  $\tau(C, h_0, \dots, h_m)$  the torsion of  $C$  with respect to the bases in chain modules constructed above and the bases  $h_0, h_1, \dots, h_m$  in homology. Since  $H_*(\partial M; Q) = 0$  the semi-linear intersection form  $H_i(M; Q) \times H_{m-i}(M; Q) \rightarrow Q$  is non-singular. Let  $v_i$  be the matrix of this form regarding the bases  $h_i$  and  $h_{m-i}$ . Put

$$d = \tau(C, h_0, h_1, \dots, h_m) \prod_{i=0}^r (\det v_i)^{-\varepsilon(i)} \in Q \setminus 0$$

where  $r = (m-1)/2$  and  $\varepsilon(i) = (-1)^{i+1}$ . It is easy to show that under a different choice of natural bases and bases  $h_0, h_1, \dots, h_m$  the element  $d$  is replaced by  $\pm gq\bar{q}d$  with  $g \in G, q \in Q \setminus 0$ . Thus the set  $\{\pm gq\bar{q}d \mid g \in Q \setminus 0\} \subset Q$  does not depend on the choice of bases. It also does not depend on the choice of triangulation in  $M$ . It is this set which is  $\omega(M)$ .

An explicit formula established in [4] enables us to calculate  $\omega(M)$  in terms of the orders of  $\mathbf{Z}[G]$ -modules  $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z})$ ,  $H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$  and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by  $J$  the image of the inclusion homomorphism  $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$  where  $r = (m-1)/2$ . Then up to multiples of type  $q\bar{q}$  with  $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord} (\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities  $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$  imply that  $H_*(\partial\tilde{M})$  and  $J$  are torsion  $\mathbf{Z}[G]$ -modules. Therefore  $\text{ord } H_i(\partial\tilde{M})$  and  $\text{ord } J$  are non-zero elements of  $\mathbf{Z}[G]$ .

We shall apply formula (4) in the case where  $M$  is the exterior of an  $n$ -component link  $K \subset S^m$  with odd  $m$ . The condition  $H_*(\partial M; Q) = 0$  is always fulfilled in this case. Here the field  $Q$  is canonically identified with the field of rational functions of  $n$  variables  $Q_n = Q(t_1, \dots, t_n)$ . Thus  $\omega(M) \subset Q_n$ . If  $m \geq 5$  then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If  $m = 3$  then there exists a unique subset  $\alpha = \alpha(K)$  of the set  $\{1, 2, \dots, n\}$  such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

### § 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module  $H$  over a (commutative) domain  $R$  we denote by  $\text{rk}_R H$  or, briefly, by  $\text{rk } H$  the integer  $\dim_Q(Q \otimes_R H)$  where  $Q = Q(R)$  denotes the field of fractions of  $R$ . For a  $R$ -linear homomorphism  $f: H \rightarrow H'$  we put  $\text{rk } f = \text{rk}_R f(H)$ . Note that if  $\bar{R}$  is the localization of  $R$  at some multiplicative system then  $Q(\bar{R}) = Q(R)$  and therefore the (exact) functor  $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$