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# ON TORRES-TYPE RELATIONS FOR THE ALEXANDER POLYNOMIALS OF LINKS

# by V. G. TURAEV

## § 1. INTRODUCTION

The classical formula of Torres [5] relates the (first) Alexander polynomial of a link K in  $S^3$  with that of the sublink of K obtained by deleting a component. The aim of the present paper is to establish a Torres-type formula for Alexander polynomials of higher-dimensional links. We also discuss analogous formulas for higher Alexander polynomials of links in  $S^3$ .

An *n*-component link in the sphere  $S^m$  is an ordered collection of *n* disjoint smooth imbedded oriented (m-2)-dimensional spheres in  $S^m$ . With each odd-dimensional link  $K \subset S^{2r+1}$  one associates a  $\Lambda_n$ -module  $H_r(\tilde{X})$ , where  $\Lambda_n$  is the Laurent polynomial ring  $\mathbb{Z}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}], X$  is the exterior of K and  $\tilde{X}$  is the maximal abelian covering of X. The module  $H_r(\tilde{X})$  algebraically gives rise to a sequence of Fitting (or determinantal) invariants  $\Delta_1(K), \Delta_2(K), ...,$  which are elements of  $\Lambda_n$  defined up to multiplication by monomials  $\pm t_1^{s_1} \dots t_n^{s_n}$  (see [1] or § 3). The polynomial  $\Delta_i(K)$  is called the *i*-th Alexander polynomial of K. The first Alexander polynomial  $\Delta_1(K)$  is also denoted by  $\Delta(K)$  and called "the Alexander polynomial of K".

THEOREM (Torres [5]). Let K be an n-component link in  $S^3$  with  $n \ge 2$  and let L be the sublink of K obtained by deleting the n-th component. Then

$$\Delta(K) (t_1, ..., t_{n-1}, 1) = \begin{cases} (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1) \Delta(L) & \text{if } n > 2\\ \frac{t_1^{l_1} - 1}{t_1 - 1} \Delta(L) & \text{if } n = 2 \end{cases}$$

where  $l_i$  denotes the linking number of the *i*-th and *n*-th components of K.

The following theorem can be considered as a high-dimensional variant of the Torres theorem.

THEOREM 1. Let K be an n-component link in  $S^m$  with odd  $m \ge 5$ . Let L be the sublink of K obtained by deleting the n-th component. Then there exists an element  $\lambda$  of  $\Lambda_{n-1}$  such that

(1) 
$$\Delta(L) = \Delta(K) (t_1, ..., t_{n-1}, 1) \cdot \lambda \overline{\lambda}.$$

Here the overbar denotes the involution of the Laurent polynomial ring  $\Lambda_{n-1}$  which sends each polynomial  $f(t_1, ..., t_{n-1})$  into  $f(t_1^{-1}, ..., t_{n-1}^{-1})$ .

It is well known that for any link  $K \subset S^m$  with odd  $m \ge 5$  the Alexander polynomial  $\Delta(K)$  is non-zero. Moreover,

$$aug(\Delta(K)) = \Delta(K)(1, 1, ..., 1) = \pm 1$$

(see [1]). This implies that  $\operatorname{aug}(\lambda) = \pm 1$  for any  $\lambda$  satisfying (1). It seems that there are no other restrictions on  $\lambda$ ; one may even guess that for any  $\Delta \in \Lambda_n$ ,  $\lambda \in \Lambda_{n-1}$  with  $\operatorname{aug}(\Delta) = \operatorname{aug}(\lambda) = \pm 1$  and  $\overline{\Delta} \doteq \Delta$ there exists a pair K, L as in Theorem 1 such that  $\Delta(K) \doteq \Delta$  and  $\Delta(L) \doteq \Delta(t_1, ..., t_{n-1}, 1)\lambda\overline{\lambda}$ . Here and below the symbol  $\doteq$  denotes the equality of Laurent polynomials up to multiplication by a monomial  $\pm t_1^{s_1} \dots t_n^{s_n}$ .

Let us call two Laurent polynomials  $\Delta, \Delta' \in \Lambda_n$  algebraically cobordant if there exist polynomials  $\lambda, \lambda' \in \Lambda_n$  such that  $\Delta\lambda\bar{\lambda} \doteq \Delta'\lambda'\bar{\lambda'}$  and aug  $(\lambda)$  $= \operatorname{aug}(\lambda') = \pm 1$ . This terminology is suggested by the fact that Alexander polynomials of (smoothly) cobordant links are algebraically cobordant (see [4]). The formula (1) enables us to calculate Alexander polynomials of all sublinks of a given link, up to algebraic cobordism. It is curious to note that if K, K' are *n*-component links in  $S^m$  with odd  $m \ge 5$  and if polynomials  $\Delta(K), \Delta(K')$  are algebraically cobordant then Theorem 1 implies that Alexander polynomials of corresponding sublinks of K, K' are algebraically cobordant to each other. This fact reflects the evident property of geometric cobordisms: corresponding sublinks of cobordant links are cobordant.

I do not know if it is possible to associate with a link K some preferred  $\lambda = \lambda(K)$  satisfying (1).

The remaining part of the Introduction is concerned with the classical links. The symbols  $K, L, n, l_1, ..., l_{n-1}$  denote the same objects as in the Torres theorem formulated above. It may well happen that some of the Alexander polynomials  $\Delta_1(K), \Delta_2(K), ...$  are equal to zero. Denote by u = u(K) the minimal integer  $u \ge 1$  such that  $\Delta_u(K) \ne 0$ . Since  $\Delta_{i+1}(K)$  divides  $\Delta_i(K)$ for all  $i, \Delta_i(K) = 0$  for i < u and  $\Delta_i(K) \ne 0$  for  $i \ge u(K)$ . In view of the Torres theorem it is natural to look for a relationship between  $\Delta_{u(K)}(K)$  and a corresponding invariant of *L*. In the case u(K) = 1we have the Torres formula, so we shall restrict ourselves to the case  $u(K) \ge 2$  (i.e. the case  $\Delta(K) = 0$ ).

The integers u(K), u(L) are related by the inequality  $u(L) \ge u(K) - 1$ (see [1] or § 4). If  $l_i \ne 0$  at least for one i = 1, ..., n - 1 then the stronger inequality holds:  $u(L) \ge u(K)$ . These inequalities suggest to relate  $\Delta_u(K)$ (where we put u = u(K)) with  $\Delta_{u-1}(L)$  and  $\Delta_u(L)$ . The following relationship between  $\Delta_u(K)$  and  $\Delta_u(L)$  was established in [4].

THEOREM ([4, Theorem 5.5.1]). If  $u = u(K) \ge 2$  then there exist an element  $\lambda$  of  $\Lambda_{n-1}$  and a subset  $\beta$  of the set  $\{1, 2, ..., n-1\}$  such that (2)  $(t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1)\Delta_u(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \overline{\lambda} \cdot \Delta_u(K) (t_1, ..., t_{n-1}, 1)$ .

Several remarks are in order. a) The non-trivial case of the Theorem is the case where at least one of the integers  $l_1, ..., l_{n-1}$  is non-zero: otherwise  $t_1^{l_1} ... t_{n-1}^{l_{n-1}} - 1 = 0$  and we may put  $\lambda = 0$ . b) Formula (2) is proved in [4] under the additional condition u(L) = u(K). However if u(L) < u(K) then we have the trivial case  $l_1 = l_2 = ... = l_{n-1} = 0$ ; if u(L) > u(K) then  $\Delta_{u(K)}(L) = 0$  and we may put  $\lambda = 0$ . c) Formula (2) combines the factors from the Torres formula, formula (1) and a new factor  $\prod (t_i-1)$ . All these factors may be non-trivial (see [4]). d) An explicit construction of the set  $\beta = \beta(K)$  is given in [4, § 5]. I do not know if there exists a preferred  $\lambda = \lambda(K)$  which satisfies (2).

The relationships between the polynomials  $\Delta_u(K)$  and  $\Delta_{u-1}(L)$  were first considered by Levine [2] in the case u = 2.

THEOREM (Levine [2]). If  $u(K) \ge 2$  then there exist an element  $\lambda \in \Lambda_{n-1}$ and a set  $\beta \subset \{1, 2, ..., n-1\}$  such that

$$\Delta(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \overline{\lambda} \cdot \Delta_2(K) (t_1, ..., t_{n-1}, 1).$$

Note that in the case u(K) > 2 the Levine's theorem is evident: if u(K) > 2 then  $u(L) \ge u(K) - 1 > 1$  so that  $\Delta(L) = \Delta_2(K) = 0$ .

The following theorem generalizes the Levine's result.

THEOREM 2. If  $u = u(K) \ge 2$  then there exist an element  $\lambda$  of  $\Lambda_{n-1}$ and a set  $\beta \subset \{1, 2, ..., n-1\}$  such that

$$\Delta_{u-1}(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \overline{\lambda} \cdot \Delta_u(K) (t_1, ..., t_{n-1}, 1).$$

The non-trivial case of Theorem 2 is the case  $l_1 = l_2 = ... = l_{n-1} = 0$ : otherwise  $u(L) \ge u$  so that  $\Delta_{u-1}(L) = 0$  and we may put  $\lambda = 0$ .

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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# $\S 2$ . Torsions of chain complexes and manifolds

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let Q be a field. If  $a = (a_1, ..., a_n)$  and  $b = (b_1, ..., b_n)$  are two bases of a Q-module then  $a_i = \sum_{j=1}^n c_{i,j}b_j$  where  $(c_{i,j})$  is a non-singular  $n \times n$ -matrix over Q; the determinant det  $(c_{i,j}) \in Q \setminus 0$  is denoted by [a/b].

Let  $C = (C_m \rightarrow \cdots \rightarrow C_0)$  be a chain Q-complex. Suppose that each Q-module  $C_i$  is finite dimensional with a preferred basis  $c_i$  and each Q-module  $H_i(C)$  also has a preferred basis  $h_i$ . (The case  $C_i = 0$  or  $H_i(C) = 0$  is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion  $\tau(C) \in Q$  as follows. For each i = 1, 2, ..., m choose a sequence  $b_i = (b_1^i, ..., b_{r_i}^i)$  of elements of  $C_i$  such that  $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), ..., \partial_{i-1}(b_{r_i}^i))$  is a basis in  $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$ . For each i = 0, 1, ..., m choose a lifting  $\tilde{h}_i$  of the basis  $h_i$  to Ker  $\partial_{i-1}$ . The combined sequence  $\partial_i(b_{i+1})\tilde{h}_ib_i$  is a basis in  $C_i$ . (It is understood that  $b_0 = \emptyset$  and  $b_{m+1} = \emptyset$ ). Put

(3) 
$$\tau(C) = \prod_{i=0}^{m} \left[ \partial_i (b_{i+1}) \tilde{h}_i b_i / c_i \right]^{\varepsilon(i)}$$

where  $\varepsilon(i) = (-1)^{i+1}$ . Clearly,  $\tau(C) \in Q \setminus 0$ . It is easy to verify that  $\tau(C)$  does not depend on the choice of  $b_i$  and  $\tilde{h_i}$ .

(Note that the torsion of C defined in Milnor's survey article [3] equals  $\pm \tau(C)^{-1} \in Q/\pm 1$  and that Milnor uses the additive notation for the multiplication in  $Q \setminus 0 = K_1(Q)$ .)

2.1.1. LEMMA (multiplicativity of torsion). Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ be a short exact sequence of m-dimensional chain complexes over a field Q.