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ON TORRES-TYPE RELATIONS FOR THE ALEXANDER POLYNOMIALS OF LINKS

by V. G. TURAEV

§ 1. INTRODUCTION

The classical formula of Torres [5] relates the (first) Alexander polynomial of a link K in S^3 with that of the sublink of K obtained by deleting a component. The aim of the present paper is to establish a Torres-type formula for Alexander polynomials of higher-dimensional links. We also discuss analogous formulas for higher Alexander polynomials of links in S^3.

An n-component link in the sphere S^m is an ordered collection of n disjoint smooth imbedded oriented $(m-2)$-dimensional spheres in S^m. With each odd-dimensional link $K \subset S^{2r+1}$ one associates a Λ_n-module $H_r(\tilde{X})$, where Λ_n is the Laurent polynomial ring $\mathbf{Z}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$, X is the exterior of K and $\tilde{X}$ is the maximal abelian covering of X. The module $H_r(\tilde{X})$ algebraically gives rise to a sequence of Fitting (or determinantal) invariants $\Delta_1(K), \Delta_2(K), ...$, which are elements of Λ_n defined up to multiplication by monomials $\pm t_1^{s_1} ... t_n^{s_n}$ (see [1] or § 3). The polynomial $\Delta_i(K)$ is called the i-th Alexander polynomial of K. The first Alexander polynomial $\Delta_1(K)$ is also denoted by $\Delta(K)$ and called "the Alexander polynomial of K".

THEOREM (Torres [5]). *Let K be an n-component link in S^3 with $n \geqslant 2$ and let L be the sublink of K obtained by deleting the n-th component. Then*

$$\Delta(K)(t_1, ..., t_{n-1}, 1) = \begin{cases} (t_1^{l_1} ... t_{n-1}^{l_{n-1}} - 1)\Delta(L) & \text{if} \quad n > 2 \\[2ex] \dfrac{t_1^{l_1} - 1}{t_1 - 1}\Delta(L) & \text{if} \quad n = 2 \end{cases}$$

where l_i denotes the linking number of the i-th and n-th components of K.

The following theorem can be considered as a high-dimensional variant of the Torres theorem.

THEOREM 1. *Let K be an n-component link in S^m with odd $m \geqslant 5$. Let L be the sublink of K obtained by deleting the n-th component. Then there exists an element λ of Λ_{n-1} such that*

$$\Delta(L) = \Delta(K)(t_1, \ldots, t_{n-1}, 1) \cdot \lambda\bar{\lambda}. \tag{1}$$

Here the overbar denotes the involution of the Laurent polynomial ring Λ_{n-1} which sends each polynomial $f(t_1, \ldots, t_{n-1})$ into $f(t_1^{-1}, \ldots, t_{n-1}^{-1})$.

It is well known that for any link $K \subset S^m$ with odd $m \geqslant 5$ the Alexander polynomial $\Delta(K)$ is non-zero. Moreover,

$$\operatorname{aug}(\Delta(K)) = \Delta(K)(1, 1, \ldots, 1) = \pm 1$$

(see [1]). This implies that $\operatorname{aug}(\lambda) = \pm 1$ for any λ satisfying (1). It seems that there are no other restrictions on λ; one may even guess that for any $\Delta \in \Lambda_n$, $\lambda \in \Lambda_{n-1}$ with $\operatorname{aug}(\Delta) = \operatorname{aug}(\lambda) = \pm 1$ and $\bar{\Delta} \doteq \Delta$ there exists a pair K, L as in Theorem 1 such that $\Delta(K) \doteq \Delta$ and $\Delta(L) \doteq \Delta(t_1, \ldots, t_{n-1}, 1)\lambda\bar{\lambda}$. Here and below the symbol $\doteq$ denotes the equality of Laurent polynomials up to multiplication by a monomial $\pm t_1^{s_1} \ldots t_n^{s_n}$.

Let us call two Laurent polynomials $\Delta, \Delta' \in \Lambda_n$ algebraically cobordant if there exist polynomials $\lambda, \lambda' \in \Lambda_n$ such that $\Delta\lambda\bar{\lambda} \doteq \Delta'\lambda'\bar{\lambda'}$ and $\operatorname{aug}(\lambda) = \operatorname{aug}(\lambda') = \pm 1$. This terminology is suggested by the fact that Alexander polynomials of (smoothly) cobordant links are algebraically cobordant (see [4]). The formula (1) enables us to calculate Alexander polynomials of all sublinks of a given link, up to algebraic cobordism. It is curious to note that if K, K' are n-component links in S^m with odd $m \geqslant 5$ and if polynomials $\Delta(K), \Delta(K')$ are algebraically cobordant then Theorem 1 implies that Alexander polynomials of corresponding sublinks of K, K' are algebraically cobordant to each other. This fact reflects the evident property of geometric cobordisms: corresponding sublinks of cobordant links are cobordant.

I do not know if it is possible to associate with a link K some preferred $\lambda = \lambda(K)$ satisfying (1).

The remaining part of the Introduction is concerned with the classical links. The symbols $K, L, n, l_1, \ldots, l_{n-1}$ denote the same objects as in the Torres theorem formulated above. It may well happen that some of the Alexander polynomials $\Delta_1(K), \Delta_2(K), \ldots$ are equal to zero. Denote by $u = u(K)$ the minimal integer $u \geqslant 1$ such that $\Delta_u(K) \neq 0$. Since $\Delta_{i+1}(K)$ divides $\Delta_i(K)$ for all i, $\Delta_i(K) = 0$ for $i < u$ and $\Delta_i(K) \neq 0$ for $i \geqslant u(K)$.

In view of the Torres theorem it is natural to look for a relationship between $\Delta_{u(K)}(K)$ and a corresponding invariant of L. In the case $u(K) = 1$ we have the Torres formula, so we shall restrict ourselves to the case $u(K) \geqslant 2$ (i.e. the case $\Delta(K) = 0$).

The integers $u(K), u(L)$ are related by the inequality $u(L) \geqslant u(K) - 1$ (see [1] or § 4). If $l_i \neq 0$ at least for one $i = 1, \ldots, n-1$ then the stronger inequality holds: $u(L) \geqslant u(K)$. These inequalities suggest to relate $\Delta_u(K)$ (where we put $u = u(K)$) with $\Delta_{u-1}(L)$ and $\Delta_u(L)$. The following relationship between $\Delta_u(K)$ and $\Delta_u(L)$ was established in [4].

THEOREM ([4, Theorem 5.5.1]). *If* $u = u(K) \geqslant 2$ *then there exist an element* λ *of* Λ_{n-1} *and a subset* β *of the set* $\{1, 2, \ldots, n-1\}$ *such that*

$$(2) \qquad (t_1^{l_1} \ldots t_{n-1}^{l_{n-1}} - 1)\Delta_u(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda\bar{\lambda} \cdot \Delta_u(K)(t_1, \ldots, t_{n-1}, 1).$$

Several remarks are in order. a) The non-trivial case of the Theorem is the case where at least one of the integers $l_1, \ldots, l_{n-1}$ is non-zero: otherwise $t_1^{l_1} \ldots t_{n-1}^{l_{n-1}} - 1 = 0$ and we may put $\lambda = 0$. b) Formula (2) is proved in [4] under the additional condition $u(L) = u(K)$. However if $u(L) < u(K)$ then we have the trivial case $l_1 = l_2 = \ldots = l_{n-1} = 0$; if $u(L) > u(K)$ then $\Delta_{u(K)}(L) = 0$ and we may put $\lambda = 0$. c) Formula (2) combines the factors from the Torres formula, formula (1) and a new factor $\prod(t_i - 1)$. All these factors may be non-trivial (see [4]). d) An explicit construction of the set $\beta = \beta(K)$ is given in [4, § 5]. I do not know if there exists a preferred $\lambda = \lambda(K)$ which satisfies (2).

The relationships between the polynomials $\Delta_u(K)$ and $\Delta_{u-1}(L)$ were first considered by Levine [2] in the case $u = 2$.

THEOREM (Levine [2]). *If* $u(K) \geqslant 2$ *then there exist an element* $\lambda \in \Lambda_{n-1}$ *and a set* $\beta \subset \{1, 2, \ldots, n-1\}$ *such that*

$$\Delta(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda\bar{\lambda} \cdot \Delta_2(K)(t_1, \ldots, t_{n-1}, 1).$$

Note that in the case $u(K) > 2$ the Levine's theorem is evident: if $u(K) > 2$ then $u(L) \geqslant u(K) - 1 > 1$ so that $\Delta(L) = \Delta_2(K) = 0$.

The following theorem generalizes the Levine's result.

THEOREM 2. *If* $u = u(K) \geqslant 2$ *then there exist an element* λ *of* Λ_{n-1} *and a set* $\beta \subset \{1, 2, \ldots, n-1\}$ *such that*

$$\Delta_{u-1}(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda\bar{\lambda} \cdot \Delta_u(K)(t_1, \ldots, t_{n-1}, 1).$$

The non-trivial case of Theorem 2 is the case $l_1 = l_2 = \dots = l_{n-1} = 0$: otherwise $u(L) \geqslant u$ so that $\Delta_{u-1}(L) = 0$ and we may put $\lambda = 0$.

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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§ 2. Torsions of chain complexes and manifolds

2.1. The torsion of a chain complex (see [3]). Let Q be a field. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are two bases of a Q-module then $a_i = \sum_{j=1}^{n} c_{i,j} b_j$ where $(c_{i,j})$ is a non-singular $n \times n$-matrix over Q; the determinant $\det(c_{i,j}) \in Q \backslash 0$ is denoted by $[a/b]$.

Let $C = (C_m \to \cdots \to C_0)$ be a chain Q-complex. Suppose that each Q-module C_i is finite dimensional with a preferred basis c_i and each Q-module $H_i(C)$ also has a preferred basis h_i. (The case $C_i = 0$ or $H_i(C) = 0$ is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion $\tau(C) \in Q$ as follows. For each $i = 1, 2, \dots, m$ choose a sequence $b_i = (b_1^i, \dots, b_{r_i}^i)$ of elements of C_i such that $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$ is a basis in $\operatorname{Im}(\partial_{i-1} \colon C_i \to C_{i-1})$. For each $i = 0, 1, \dots, m$ choose a lifting $\tilde{h}_i$ of the basis h_i to $\operatorname{Ker} \partial_{i-1}$. The combined sequence $\partial_i(b_{i+1})\tilde{h}_i b_i$ is a basis in C_i. (It is understood that $b_0 = \emptyset$ and $b_{m+1} = \emptyset$). Put

$$\tau(C) = \prod_{i=0}^{m} [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)} \tag{3}$$

where $\varepsilon(i) = (-1)^{i+1}$. Clearly, $\tau(C) \in Q \backslash 0$. It is easy to verify that $\tau(C)$ does not depend on the choice of b_i and $\tilde{h}_i$.

(Note that the torsion of C defined in Milnor's survey article [3] equals $\pm \tau(C)^{-1} \in Q/\pm 1$ and that Milnor uses the additive notation for the multiplication in $Q \backslash 0 = K_1(Q)$.)

2.1.1. Lemma (multiplicativity of torsion). *Let* $0 \to C' \to C \to C'' \to 0$ *be a short exact sequence of m-dimensional chain complexes over a field* Q.