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# ON TORRES-TYPE RELATIONS FOR THE ALEXANDER POLYNOMIALS OF LINKS

by V. G. TURAEV

## § 1. INTRODUCTION

The classical formula of Torres [5] relates the (first) Alexander polynomial of a link  $K$  in  $S^3$  with that of the sublink of  $K$  obtained by deleting a component. The aim of the present paper is to establish a Torres-type formula for Alexander polynomials of higher-dimensional links. We also discuss analogous formulas for higher Alexander polynomials of links in  $S^3$ .

An  $n$ -component link in the sphere  $S^m$  is an ordered collection of  $n$  disjoint smooth imbedded oriented  $(m-2)$ -dimensional spheres in  $S^m$ . With each odd-dimensional link  $K \subset S^{2r+1}$  one associates a  $\Lambda_n$ -module  $H_r(\tilde{X})$ , where  $\Lambda_n$  is the Laurent polynomial ring  $\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ ,  $X$  is the exterior of  $K$  and  $\tilde{X}$  is the maximal abelian covering of  $X$ . The module  $H_r(\tilde{X})$  algebraically gives rise to a sequence of Fitting (or determinantal) invariants  $\Delta_1(K), \Delta_2(K), \dots$ , which are elements of  $\Lambda_n$  defined up to multiplication by monomials  $\pm t_1^{s_1} \dots t_n^{s_n}$  (see [1] or § 3). The polynomial  $\Delta_i(K)$  is called the  $i$ -th Alexander polynomial of  $K$ . The first Alexander polynomial  $\Delta_1(K)$  is also denoted by  $\Delta(K)$  and called "the Alexander polynomial of  $K$ ".

**THEOREM (Torres [5]).** *Let  $K$  be an  $n$ -component link in  $S^3$  with  $n \geq 2$  and let  $L$  be the sublink of  $K$  obtained by deleting the  $n$ -th component. Then*

$$\Delta(K)(t_1, \dots, t_{n-1}, 1) = \begin{cases} (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1)\Delta(L) & \text{if } n > 2 \\ \frac{t_1^{l_1} - 1}{t_1 - 1} \Delta(L) & \text{if } n = 2 \end{cases}$$

where  $l_i$  denotes the linking number of the  $i$ -th and  $n$ -th components of  $K$ .

The following theorem can be considered as a high-dimensional variant of the Torres theorem.

THEOREM 1. Let  $K$  be an  $n$ -component link in  $S^m$  with odd  $m \geq 5$ . Let  $L$  be the sublink of  $K$  obtained by deleting the  $n$ -th component. Then there exists an element  $\lambda$  of  $\Lambda_{n-1}$  such that

$$(1) \quad \Delta(L) = \Delta(K)(t_1, \dots, t_{n-1}, 1) \cdot \lambda \bar{\lambda}.$$

Here the overbar denotes the involution of the Laurent polynomial ring  $\Lambda_{n-1}$  which sends each polynomial  $f(t_1, \dots, t_{n-1})$  into  $f(t_1^{-1}, \dots, t_{n-1}^{-1})$ .

It is well known that for any link  $K \subset S^m$  with odd  $m \geq 5$  the Alexander polynomial  $\Delta(K)$  is non-zero. Moreover,

$$\text{aug}(\Delta(K)) = \Delta(K)(1, 1, \dots, 1) = \pm 1$$

(see [1]). This implies that  $\text{aug}(\lambda) = \pm 1$  for any  $\lambda$  satisfying (1). It seems that there are no other restrictions on  $\lambda$ ; one may even guess that for any  $\Delta \in \Lambda_n$ ,  $\lambda \in \Lambda_{n-1}$  with  $\text{aug}(\Delta) = \text{aug}(\lambda) = \pm 1$  and  $\bar{\Delta} \doteq \Delta$  there exists a pair  $K, L$  as in Theorem 1 such that  $\Delta(K) \doteq \Delta$  and  $\Delta(L) \doteq \Delta(t_1, \dots, t_{n-1}, 1)\lambda\bar{\lambda}$ . Here and below the symbol  $\doteq$  denotes the equality of Laurent polynomials up to multiplication by a monomial  $\pm t_1^{s_1} \dots t_n^{s_n}$ .

Let us call two Laurent polynomials  $\Delta, \Delta' \in \Lambda_n$  algebraically cobordant if there exist polynomials  $\lambda, \lambda' \in \Lambda_n$  such that  $\Delta\lambda\bar{\lambda} \doteq \Delta'\lambda'\bar{\lambda}'$  and  $\text{aug}(\lambda) = \text{aug}(\lambda') = \pm 1$ . This terminology is suggested by the fact that Alexander polynomials of (smoothly) cobordant links are algebraically cobordant (see [4]). The formula (1) enables us to calculate Alexander polynomials of all sublinks of a given link, up to algebraic cobordism. It is curious to note that if  $K, K'$  are  $n$ -component links in  $S^m$  with odd  $m \geq 5$  and if polynomials  $\Delta(K), \Delta(K')$  are algebraically cobordant then Theorem 1 implies that Alexander polynomials of corresponding sublinks of  $K, K'$  are algebraically cobordant to each other. This fact reflects the evident property of geometric cobordisms: corresponding sublinks of cobordant links are cobordant.

I do not know if it is possible to associate with a link  $K$  some preferred  $\lambda = \lambda(K)$  satisfying (1).

The remaining part of the Introduction is concerned with the classical links. The symbols  $K, L, n, l_1, \dots, l_{n-1}$  denote the same objects as in the Torres theorem formulated above. It may well happen that some of the Alexander polynomials  $\Delta_1(K), \Delta_2(K), \dots$  are equal to zero. Denote by  $u = u(K)$  the minimal integer  $u \geq 1$  such that  $\Delta_u(K) \neq 0$ . Since  $\Delta_{i+1}(K)$  divides  $\Delta_i(K)$  for all  $i$ ,  $\Delta_i(K) = 0$  for  $i < u$  and  $\Delta_i(K) \neq 0$  for  $i \geq u(K)$ .

In view of the Torres theorem it is natural to look for a relationship between  $\Delta_{u(K)}(K)$  and a corresponding invariant of  $L$ . In the case  $u(K) = 1$  we have the Torres formula, so we shall restrict ourselves to the case  $u(K) \geq 2$  (i.e. the case  $\Delta(K) = 0$ ).

The integers  $u(K), u(L)$  are related by the inequality  $u(L) \geq u(K) - 1$  (see [1] or § 4). If  $l_i \neq 0$  at least for one  $i = 1, \dots, n-1$  then the stronger inequality holds:  $u(L) \geq u(K)$ . These inequalities suggest to relate  $\Delta_u(K)$  (where we put  $u = u(K)$ ) with  $\Delta_{u-1}(L)$  and  $\Delta_u(L)$ . The following relationship between  $\Delta_u(K)$  and  $\Delta_u(L)$  was established in [4].

THEOREM ([4, Theorem 5.5.1]). *If  $u = u(K) \geq 2$  then there exist an element  $\lambda$  of  $\Lambda_{n-1}$  and a subset  $\beta$  of the set  $\{1, 2, \dots, n-1\}$  such that*

$$(2) \quad (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1) \Delta_u(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$

Several remarks are in order. a) The non-trivial case of the Theorem is the case where at least one of the integers  $l_1, \dots, l_{n-1}$  is non-zero: otherwise  $t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1 = 0$  and we may put  $\lambda = 0$ . b) Formula (2) is proved in [4] under the additional condition  $u(L) = u(K)$ . However if  $u(L) < u(K)$  then we have the trivial case  $l_1 = l_2 = \dots = l_{n-1} = 0$ ; if  $u(L) > u(K)$  then  $\Delta_{u(K)}(L) = 0$  and we may put  $\lambda = 0$ . c) Formula (2) combines the factors from the Torres formula, formula (1) and a new factor  $\prod (t_i - 1)$ . All these factors may be non-trivial (see [4]). d) An explicit construction of the set  $\beta = \beta(K)$  is given in [4, § 5]. I do not know if there exists a preferred  $\lambda = \lambda(K)$  which satisfies (2).

The relationships between the polynomials  $\Delta_u(K)$  and  $\Delta_{u-1}(L)$  were first considered by Levine [2] in the case  $u = 2$ .

THEOREM (Levine [2]). *If  $u(K) \geq 2$  then there exist an element  $\lambda \in \Lambda_{n-1}$  and a set  $\beta \subset \{1, 2, \dots, n-1\}$  such that*

$$\Delta(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_2(K) (t_1, \dots, t_{n-1}, 1).$$

Note that in the case  $u(K) > 2$  the Levine's theorem is evident: if  $u(K) > 2$  then  $u(L) \geq u(K) - 1 > 1$  so that  $\Delta(L) = \Delta_2(K) = 0$ .

The following theorem generalizes the Levine's result.

THEOREM 2. *If  $u = u(K) \geq 2$  then there exist an element  $\lambda$  of  $\Lambda_{n-1}$  and a set  $\beta \subset \{1, 2, \dots, n-1\}$  such that*

$$\Delta_{u-1}(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$



The non-trivial case of Theorem 2 is the case  $l_1 = l_2 = \dots = l_{n-1} = 0$ : otherwise  $u(L) \geq u$  so that  $\Delta_{u-1}(L) = 0$  and we may put  $\lambda = 0$ .

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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## § 2. TORSIONS OF CHAIN COMPLEXES AND MANIFOLDS

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let  $Q$  be a field. If  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  are two bases of a  $Q$ -module then  $a_i = \sum_{j=1}^n c_{i,j} b_j$  where  $(c_{i,j})$  is a non-singular  $n \times n$ -matrix over  $Q$ ; the determinant  $\det(c_{i,j}) \in Q \setminus 0$  is denoted by  $[a/b]$ .

Let  $C = (C_m \rightarrow \dots \rightarrow C_0)$  be a chain  $Q$ -complex. Suppose that each  $Q$ -module  $C_i$  is finite dimensional with a preferred basis  $c_i$  and each  $Q$ -module  $H_i(C)$  also has a preferred basis  $h_i$ . (The case  $C_i = 0$  or  $H_i(C) = 0$  is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion  $\tau(C) \in Q$  as follows. For each  $i = 1, 2, \dots, m$  choose a sequence  $b_i = (b_1^i, \dots, b_{r_i}^i)$  of elements of  $C_i$  such that  $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$  is a basis in  $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$ . For each  $i = 0, 1, \dots, m$  choose a lifting  $\tilde{h}_i$  of the basis  $h_i$  to  $\text{Ker } \partial_{i-1}$ . The combined sequence  $\partial_i(b_{i+1})\tilde{h}_i b_i$  is a basis in  $C_i$ . (It is understood that  $b_0 = \emptyset$  and  $b_{m+1} = \emptyset$ ). Put

$$(3) \quad \tau(C) = \prod_{i=0}^m [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where  $\varepsilon(i) = (-1)^{i+1}$ . Clearly,  $\tau(C) \in Q \setminus 0$ . It is easy to verify that  $\tau(C)$  does not depend on the choice of  $b_i$  and  $\tilde{h}_i$ .

(Note that the torsion of  $C$  defined in Milnor's survey article [3] equals  $\pm \tau(C)^{-1} \in Q/\pm 1$  and that Milnor uses the additive notation for the multiplication in  $Q \setminus 0 = K_1(Q)$ .)

2.1.1. LEMMA (multiplicativity of torsion). Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  be a short exact sequence of  $m$ -dimensional chain complexes over a field  $Q$ .

Suppose that for all  $i = 0, 1, \dots, m$  the modules  $C_i, C'_i, C''_i$  are provided with preferred bases  $c'_i, c_i, c''_i$  which are compatible, in the sense that  $[c'_i c''_i / c_i] = \pm 1$ . Suppose that for all  $i = 0, 1, \dots, m$  the homology modules  $H_i(C), H_i(C'), H_i(C'')$  are provided with preferred bases. Let  $\mathcal{H}$  be the homology sequence of the sequence  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ :

$$\mathcal{H} = (H_m(C') \rightarrow H_m(C) \rightarrow \dots \rightarrow H_0(C) \rightarrow H_0(C'')).$$

Consider  $\mathcal{H}$  as an acyclic based chain complex over  $Q$ . Then  $\tau(C) = \pm \tau(C')\tau(C'')\tau(\mathcal{H})$ .

For a proof see [3].

2.2. THE TORSION  $\omega$ . Let  $M$  be an orientable compact smooth manifold of odd dimension  $m$  with  $\text{rg } H_1(M) \geq 1$ . Denote the free abelian group  $H_1(M)/\text{Tors } H_1(M)$  by  $G$ . Denote the fraction field of the group ring  $\mathbb{Z}[G]$  by  $Q$ . Provide  $Q$  with the involution  $q \mapsto \bar{q}$  which sends  $g \in G$  to  $g^{-1}$ . The field  $Q$  defines via the natural homomorphism  $\mathbb{Z}[\pi_1(M)] \rightarrow Q$  a system of local coefficients on  $M$ . We shall denote this system by the same symbol  $Q$ . Assume that  $H_*(\partial M; Q) = 0$ . In this setting one can consider a torsion-type invariant  $\omega(M)$  of  $M$  which is "an element of  $Q \setminus 0$ " defined up to multiplication by  $\pm gq\bar{q}$  with  $g \in G$  and  $q \in Q \setminus 0$  (see [4]).

Recall the definition of  $\omega(M)$  given in [4, § 5]. Let  $\tilde{M} \rightarrow M$  be the regular covering of  $M$  corresponding to the kernel of the natural homomorphism  $\pi_1(M) \rightarrow G$ . Fix a  $C^1$ -triangulation of  $M$  and the induced  $G$ -equivariant triangulation of  $\tilde{M}$ . Choose over each simplex of the (fixed) triangulation of  $M$  a simplex of the triangulation of  $\tilde{M}$ . These simplices in  $\tilde{M}$  being arbitrarily oriented and ordered determine "natural" bases of the modules of the simplicial chain  $\mathbb{Z}[G]$ -complex  $C_*(\tilde{M}; \mathbb{Z})$ . These bases induce "natural"  $Q$ -bases in the chain  $Q$ -complex

$$C = Q \otimes_{\mathbb{Z}[G]} C_*(\tilde{M}; \mathbb{Z}).$$

For all  $i = 0, 1, \dots, m$  choose an arbitrary  $Q$ -basis  $h_i$  in  $H_i(M; Q) = H_i(C)$ . Denote by  $\tau(C, h_0, \dots, h_m)$  the torsion of  $C$  with respect to the bases in chain modules constructed above and the bases  $h_0, h_1, \dots, h_m$  in homology. Since  $H_*(\partial M; Q) = 0$  the semi-linear intersection form  $H_i(M; Q) \times H_{m-i}(M; Q) \rightarrow Q$  is non-singular. Let  $v_i$  be the matrix of this form regarding the bases  $h_i$  and  $h_{m-i}$ . Put

$$d = \tau(C, h_0, h_1, \dots, h_m) \prod_{i=0}^r (\det v_i)^{-\varepsilon(i)} \in Q \setminus 0$$

where  $r = (m-1)/2$  and  $\varepsilon(i) = (-1)^{i+1}$ . It is easy to show that under a different choice of natural bases and bases  $h_0, h_1, \dots, h_m$  the element  $d$  is replaced by  $\pm gq\bar{q}d$  with  $g \in G, q \in Q \setminus 0$ . Thus the set  $\{\pm gq\bar{q}d \mid g \in G, q \in Q \setminus 0\} \subset Q$  does not depend on the choice of bases. It also does not depend on the choice of triangulation in  $M$ . It is this set which is  $\omega(M)$ .

An explicit formula established in [4] enables us to calculate  $\omega(M)$  in terms of the orders of  $\mathbf{Z}[G]$ -modules  $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z})$ ,  $H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$  and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by  $J$  the image of the inclusion homomorphism  $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$  where  $r = (m-1)/2$ . Then up to multiples of type  $q\bar{q}$  with  $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord}(\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities  $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$  imply that  $H_*(\partial\tilde{M})$  and  $J$  are torsion  $\mathbf{Z}[G]$ -modules. Therefore  $\text{ord } H_i(\partial\tilde{M})$  and  $\text{ord } J$  are non-zero elements of  $\mathbf{Z}[G]$ .

We shall apply formula (4) in the case where  $M$  is the exterior of an  $n$ -component link  $K \subset S^m$  with odd  $m$ . The condition  $H_*(\partial M; Q) = 0$  is always fulfilled in this case. Here the field  $Q$  is canonically identified with the field of rational functions of  $n$  variables  $Q_n = Q(t_1, \dots, t_n)$ . Thus  $\omega(M) \subset Q_n$ . If  $m \geq 5$  then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If  $m = 3$  then there exists a unique subset  $\alpha = \alpha(K)$  of the set  $\{1, 2, \dots, n\}$  such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

### § 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module  $H$  over a (commutative) domain  $R$  we denote by  $\text{rk}_R H$  or, briefly, by  $\text{rk } H$  the integer  $\dim_Q(Q \otimes_R H)$  where  $Q = Q(R)$  denotes the field of fractions of  $R$ . For a  $R$ -linear homomorphism  $f: H \rightarrow H'$  we put  $\text{rk } f = \text{rk}_R f(H)$ . Note that if  $\bar{R}$  is the localization of  $R$  at some multiplicative system then  $Q(\bar{R}) = Q(R)$  and therefore the (exact) functor  $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$

preserves the ranks of modules and homomorphisms. If  $H, H'$  are finitely generated free  $R$ -modules and if  $A$  is the matrix of a  $R$ -homomorphism  $H \rightarrow H'$  with respect to some bases then  $\text{rk } f = \text{rk } A$  where  $\text{rk } A$  is the maximal integer  $r$  such that some  $r \times r$ -minor of  $A$  is non-zero.

If  $R$  is a unique factorization domain with 1 and if  $A$  is a matrix with  $n < \infty$  columns and possibly infinite number of rows then  $\Delta_i(A)$  denotes the greatest common divisor of the  $(n-i+1) \times (n-i+1)$ -minors of  $A$ . Here  $i = 1, 2, \dots$  and  $\Delta_i(A)$  is an element of  $R$  defined up to a unit multiple. If  $H$  is a finitely generated module over  $R$  and  $A$  is a presentation matrix of  $H$  then  $\Delta_i(A)$  depends only on  $H$  and  $i$ ; one defines  $\Delta_i(H) = \Delta_i(A)$ . Clearly  $\Delta_i(H) = 0$  for  $i \leq \text{rg } H = n - \text{rg } A$  and  $\Delta_i(H) \neq 0$  for  $i > \text{rg } H$ . The invariant  $\Delta_1(H)$  is denoted also by  $\text{ord } H$ ; it is called the order of  $H$ . It is clear that  $\text{ord } H \neq 0$  iff  $H = \text{Tors}_R H$ . For proofs and further information see [1].

Recall, finally, that a local ring is a domain  $K$  which has a unique maximal (proper) ideal. The quotient of  $K$  by this ideal is a field which we shall call "the field associated to  $K$ ".

**3.2. LEMMA.** *Let  $R, R'$  be (commutative) domains with 1 and let  $\varphi: R \rightarrow R'$  be a ring homomorphism. Let  $C = (\cdots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \cdots)$  be a finitely generated free chain complex over  $R$  and let  $C'$  be the chain  $R'$ -complex  $R' \otimes_R C$ . Then: (i)  $\text{rk}_{R'} H_i(C') \geq \text{rk}_R H_i(C)$  and  $\text{rk } \partial'_i \leq \text{rk } \partial_i$  for all  $i$  where  $\partial_i, \partial'_i$  are the boundary homomorphisms  $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$ ; (ii) if  $\text{rk } H_i(C') = \text{rk } H_i(C)$  for some  $i$  then  $\text{rk } \partial'_j = \text{rk } \partial_j$  for  $j = i, i+1$ ; (iii) if  $R, R'$  are unique factorization Noetherian domains and if  $\text{rk } H_i(C') = \text{rk } H_i(C)$  then  $\varphi(\text{ord}(\text{Tors}_R H_i(C)))$  divides  $\text{ord}(\text{Tors}_{R'} H_i(C'))$ .*

*Proof.* Let  $n = \text{rk } C_i$ . Let  $A = (a_{p,q})$ ,  $1 \leq q \leq n$ ,  $1 \leq p$ , be the matrix of  $\partial_i$  with respect to some bases in  $C_i, C_{i+1}$ . Then  $A' = (\varphi(a_{p,q}))$  is the matrix of  $\partial'_i$  with respect to the induced bases in  $C'_i, C'_{i+1}$ . It is evident that  $\text{rk } \partial'_i = \text{rk } A' \leq \text{rk } A = \text{rk } \partial_i$ . Therefore

$$\text{rk } H_i(C') = n - \text{rk } \partial'_i - \text{rk } \partial'_{i+1} \geq n - \text{rk } \partial_i - \text{rk } \partial_{i+1} = \text{rk } H_i(C).$$

These inequalities imply (i) and (ii).

Put  $r = n - \text{rk } A + 1$  and denote the  $R$ -module  $C_i/\text{Im } \partial_i$  by  $J$ . Since  $A$  is a presentation matrix of  $J$  we have  $\text{ord}(\text{Tors}_R J) = \Delta_r(A)$  (see [1, p. 31]). From the exact sequence  $0 \rightarrow H_i(C) \rightarrow J \rightarrow C_{i-1}$  we obtain that  $\text{Tors } J = \text{Tors } H_i(C)$ . Thus  $\text{ord}(\text{Tors } H_i(C)) = \Delta_r(A)$ . Analogously  $\text{ord}(\text{Tors } H_i(C')) = \Delta_{r'}(A')$  where  $r' = n - \text{rk } A' + 1$ . If  $\text{rk } H_i(C) = \text{rk } H_i(C')$  then  $\text{rk } A = \text{rk } A'$  and therefore  $r = r'$ . It is evident that  $\varphi(\Delta_j(A))$  divides  $\Delta_j(A')$  for all  $j$ . This implies (iii).

3.3. LEMMA. Let  $R$  be a local ring and  $F$  be the associated field. Let  $f: C_1 \rightarrow C_0$  be a  $R$ -homomorphism of finitely generated free  $R$ -modules and let  $\bar{f}: F \otimes_R C_1 \rightarrow F \otimes_R C_0$  be the induced  $F$ -homomorphism. If  $\text{rk } f = \text{rk } \bar{f}$  then with respect to some bases in  $C_1, C_0$  the homomorphism  $f$  is presented by the matrix  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$  where  $E$  is the unit matrix of order  $\text{rk } f$ .

*Proof.* Since  $F$  is a field we can choose bases  $d_0, d_1$  respectively in  $F \otimes_R C_0, F \otimes_R C_1$  so that the matrix of  $\bar{f}$  regarding these bases has the form  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ . Let  $\mathcal{D}_i$  be a lifting of  $d_i$  to  $C_i, i = 1, 2$ . Here  $\mathcal{D}_i$  is a sequence of  $\text{rg } C_i$  elements of  $C_i$ . In view of Nakayama's lemma  $\mathcal{D}_i$  generate  $C_i$ . This implies that  $\mathcal{D}_i$  generates the  $(\text{rg } C_i)$ -dimensional vector space  $Q(R) \otimes_R C_i$  over the field  $Q(R)$ . Therefore, the elements of the sequence  $\mathcal{D}_i$  are linearly independent over  $Q(R)$  and, hence, over  $R$ . Thus  $\mathcal{D}_i$  is a basis of  $C_i$  for  $i = 0, 1$ . The matrix of  $f$  with respect to bases  $\mathcal{D}_0, \mathcal{D}_1$  has the form  $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$  where  $U, X, Y, Z$  are matrices over the maximal ideal  $u$  of  $R$ . Note that  $\det(E+U) = 1 \pmod{u}$ . Since all elements of  $R \setminus u$  are invertible in  $R$  the square matrix  $E+U$  is invertible over  $R$ . Therefore we can choose bases in  $C_0, C_1$  so that the corresponding matrix of  $f$  equals  $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$ . Since  $\text{rk } f = \text{rk } \bar{f} = \text{rk } E, Y' = 0$ .

3.4. LEMMA. Let  $R$  be a local ring and  $F$  be the associated field. Let  $C = (\cdots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \cdots)$  be a finitely generated free chain complex over  $R$ . Let  $C'$  be the chain  $F$ -complex  $F \otimes_R C$ . Let  $\partial_i, \partial'_i$  be the boundary homomorphisms  $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$ . If  $\text{rk}_R H_i(C) = \text{rk}_F H_i(C')$  for some  $i$  then:  $H_i(C), \text{Im } \partial_{i+1}, \text{Im } \partial_i$  are free  $R$ -modules and  $C_i = \text{Im } \partial_{i+1} \oplus H_i(C) \oplus \text{Im } \partial_i$ ; the projection  $C \rightarrow C'$  induces  $F$ -isomorphisms  $F \otimes_R H_i(C) \rightarrow H_i(C'), F \otimes_R \text{Im } \partial_j \rightarrow \text{Im } \partial'_j$  with  $j = i, i+1$ .

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

#### § 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by  $Q_n$  the fraction field of the ring  $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ . Denote by  $Q_n^0$  the subring of  $Q_n$  which consists of rational functions  $fg^{-1}$  with  $f, g \in \Lambda_n$  and  $g \notin (t_n - 1)\Lambda_n$  (so that

$g(t_1, \dots, t_{n-1}, 1) \neq 0$ ). The homomorphism  $f \mapsto f(t_1, \dots, t_{n-1}, 1): \Lambda_n \rightarrow \Lambda_{n-1}$  uniquely extends to a ring homomorphism  $Q_n^0 \rightarrow Q_{n-1}$  which is denoted by  $\varphi$ .

Denote by  $X$  the exterior of  $K$  and by  $Y$  the exterior of  $L$ .

We shall prove the following two statements.

(4.1.1).  $\varphi(\Delta(K)) = \Delta(K)(t_1, \dots, t_{n-1}, 1)$  divides  $\Delta(L)$  in  $\Lambda_{n-1}$ .

(4.1.2). There exists a representative  $\omega$  of the torsion  $\omega(X) \subset Q_n$  such that  $(t_n - 1)\omega \in Q_n^0$  and  $\varphi((t_n - 1)\omega)$  represents  $\omega(Y) \subset Q_{n-1}$ .

Let us show first that these two statements imply the Theorem. Let  $\omega$  be the element of  $Q_n$  produced by (4.1.2). Put  $\pi = \prod_{i=1}^{n-1} (t_i - 1)$ . According to the results formulated in Sec. 2.2 the product  $(t_n - 1)\pi \cdot \Delta(K)$  represents  $\omega(X)$ . Thus

$$\omega \doteq \frac{f\bar{f}}{g\bar{g}}(t_n - 1)\pi\Delta(K)$$

where  $f, g \in \Lambda_n \setminus 0$ . We may assume that  $f\bar{f}$  and  $g\bar{g}$  are relatively prime. If  $t_n - 1$  does not divide  $g$  then  $\omega \in Q_n^0$  and  $\varphi((t_n - 1)\omega) = 0$  which contradicts to the inclusion  $\varphi((t_n - 1)\omega) \in \omega(Y)$ . Thus  $g = (t_n - 1)h$  with  $h \in \Lambda_n$ . In view of (4.1.1),  $\varphi(\Delta(K)) \neq 0$ , i.e.  $t_n - 1$  does not divide  $\Delta(K)$ . If  $\varphi(h) = 0$  then  $(t_n - 1)^2$  divides  $g$  which obviously contradicts the inclusion  $(t_n - 1)\omega \in Q_n^0$ . Thus  $\varphi(h) \neq 0$ . We have

$$h\bar{h}(t_n - 1)\omega \doteq f\bar{f}\pi\Delta(K).$$

Since  $\varphi(h\bar{h}(t_n - 1)\omega) \neq 0$  we have  $\varphi(f) \neq 0$ . This implies that  $\pi \cdot \varphi(\Delta(K)) \doteq q\bar{q}\varphi((t_n - 1)\omega)$  where  $q = \varphi(h)/\varphi(f)$ . Thus  $\pi\varphi(\Delta(K))$  represents  $\omega(Y)$ . Since  $\pi\Delta(L) \in \omega(Y)$  we have

$$\varphi(\Delta(K))\lambda\bar{\lambda} = \Delta(L)\mu\bar{\mu}$$

with non-zero  $\lambda, \mu \in \Lambda_{n-1}$ . We may assume that  $\lambda\bar{\lambda}$  and  $\mu\bar{\mu}$  are relatively prime. Since  $\varphi(\Delta(K))$  divides  $\Delta(L)$  we immediately obtain  $\mu\bar{\mu} = 1$ . Thus,  $\Delta(L) = \varphi(\Delta(K))\lambda\bar{\lambda}$ .

Let us prove (4.1.1) and (4.1.2). We may assume that  $X \subset Y$  and that  $Y \setminus X$  is the interior of the regular neighborhood  $U \subset Y$  of the  $n$ -th component of  $K$  in  $Y$ . Let  $p: \tilde{X} \rightarrow X$  and  $q: \tilde{Y} \rightarrow Y$  be the maximal abelian coverings with the groups of covering transformations respectively  $H_1(X) \approx \mathbf{Z}^n$  (generators  $t_1, \dots, t_n$ ) and  $H_1(Y) \approx \mathbf{Z}^{n-1}$  (generators  $t_1, \dots, t_{n-1}$ ). It is clear that  $p$  is the composition of an infinite cyclic covering  $\tilde{X} \rightarrow q^{-1}(X)$  and the covering  $q: q^{-1}(X) \rightarrow X$ .

Fix a  $C^1$ -triangulation of  $Y$  so that  $X$  and  $U$  are simplicial subcomplexes of  $Y$ . Fix also the induced equivariant triangulations in  $\tilde{X}$  and  $\tilde{Y}$ .

The ring  $\Lambda_{n-1}$  determines via the natural homomorphism  $\mathbf{Z}[\pi_1(Y)] \rightarrow \mathbf{Z}[H_1 Y] = \Lambda_{n-1}$  a system of local coefficients on  $Y$  which we denote by the same symbol  $\Lambda_{n-1}$ . According to definitions, for any simplicial subsets  $A \supset B$  of  $Y$  the  $\Lambda_{n-1}$ -module  $H_*(A, B; \Lambda_{n-1})$  equals  $H_*(C(q^{-1}(A), q^{-1}(B); \mathbf{Z}))$ . Here the simplicial chain complex  $C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$  is a finitely generated free  $\Lambda_{n-1}$ -complex. Analogously  $\Lambda_n$  defines a system of local coefficients on  $X$  and for simplicial subsets  $A \supset B$  of  $X$  the  $\Lambda_n$ -module  $H_*(A, B; \Lambda_n)$  equals  $H_*(C(p^{-1}(A), p^{-1}(B); \mathbf{Z}))$ . Note that

$$\Lambda_{n-1} \otimes_{\Lambda_n} C_*(p^{-1}(A), p^{-1}(B); \mathbf{Z}) = C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$$

where  $\Lambda_n$  acts on  $\Lambda_{n-1}$  via  $\phi$ .

*Claim 1.* For  $i \neq 1, m-1$ ,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = 0.$$

For  $i = 1, m-1$ ,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = n-1; \quad \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = n-2.$$

*Proof of Claim 1.* We shall compute the rank of  $H_i(X; \Lambda_n)$ ; modules  $H_i(X; \Lambda_{n-1})$  and  $H_i(Y; \Lambda_{n-1})$  can be treated similarly.

Denote by  $V$  a wedge of  $n$  circles in  $X$  such that the inclusion homomorphism  $H_1(V; \mathbf{Z}) \rightarrow H_1(X; \mathbf{Z}) = \mathbf{Z}^n$  is bijective. Then  $H_i(X, V; \mathbf{Z}) = 0$  for  $i \leq m-2$ . Therefore an application of Lemma 3.2(i) to complexes  $C_*(\tilde{X}, p^{-1}(V); \mathbf{Z})$  and  $C_*(X, V; \mathbf{Z})$  gives that  $\mathrm{rk}_{\Lambda_n} H_i(X, V; \Lambda_n) = 0$  for  $i \leq m-2$ . This implies that  $\mathrm{rk} H_i(X; \Lambda_n) = \mathrm{rk} H_i(V; \Lambda_n)$  for  $i \leq m-3$  and that  $\mathrm{rk} H_{m-2}(X; \Lambda_n) \leq \mathrm{rk} H_{m-2}(V; \Lambda_n)$ . The rank of  $H_i(V; \Lambda_n)$  can be computed directly: It is equal to 0 if  $i \neq 1$  and to  $n-1$  if  $i = 1$ . Thus the rank of  $H_i(X; \Lambda_n)$  equals 0 if  $i \neq 1, m-1$  and equals  $n-1$  if  $i = 1$ . The equality  $\mathrm{rk} H_{m-1}(X; \Lambda_n) = n-1$  follows from duality or from the equalities

$$\sum_{i=0}^m (-1)^i \mathrm{rk} H_i(X; \Lambda_n) = \chi(X) = 0.$$

*Claim 2.* The exact homology sequence of  $(Y, X)$  with coefficients in  $\Lambda_{n-1}$  splits into short exact sequences



$$\begin{aligned}
0 &\rightarrow H_m(Y, X; \Lambda_{n-1}) \rightarrow H_{m-1}(X; \Lambda_{n-1}) \rightarrow H_{m-1}(Y; \Lambda_{n-1}) \rightarrow 0, \\
0 &\rightarrow H_i(X; \Lambda_{n-1}) \xrightarrow{\approx} H_i(Y; \Lambda_{n-1}) \rightarrow 0, \quad (i \neq 1, m-1) \\
0 &\rightarrow H_2(Y, X; \Lambda_{n-1}) \xrightarrow{\partial_1} H_1(X; \Lambda_{n-1}) \rightarrow H_1(Y; \Lambda_{n-1}) \rightarrow 0.
\end{aligned}$$

*Proof of Claim 2.* Clearly,  $H_i(Y, X; \Lambda_{n-1}) = H_i(U, \partial U; \Lambda_{n-1}) = 0$  for  $i \neq 2, m$ . Therefore the only thing to prove is the injectivity of  $\partial_1$ . According to Claim 1  $\text{rk } H_1(X; \Lambda_{n-1}) = n - 1$  and  $\text{rk } H_1(Y; \Lambda_{n-1}) = n - 2$ . Since  $H_2(Y, X; \Lambda_{n-1}) = \Lambda_{n-1}$  we see that  $\partial_1$  is injective.

*Proof of (4.1.1).* In view of the equalities  $\text{rg } H_i(X; \Lambda_n) = \text{rg } H_i(X; \Lambda_{n-1})$ ,  $i = 0, 1, \dots$  we may apply Lemma 3.2 (iii) to the chain complexes  $C_*(\tilde{X}; \mathbf{Z})$  and  $C_*(q^{-1}(X); \mathbf{Z})$  respectively over  $\Lambda_n$  and  $\Lambda_{n-1}$ . Since  $m - 1 > r > 1$  Claims 1, 2 show that  $H_r(X; \Lambda_n)$  and  $H_r(X; \Lambda_{n-1})$  are torsion modules respectively over  $\Lambda_n$  and  $\Lambda_{n-1}$  and  $H_r(X, \Lambda_{n-1}) = H_r(Y; \Lambda_{n-1})$ . By definition  $\Delta(K) = \text{ord } H_r(X; \Lambda_n)$  and  $\Delta(L) = \text{ord } H_r(Y; \Lambda_{n-1}) = \text{ord } H_r(X; \Lambda_{n-1})$ . Lemma 3.2 (iii) directly implies that  $\phi(\Delta(K))$  divides  $\Delta(L)$ .

It remains to prove Statement (4.1.2) which is, of course, the core of Theorem 1. For simplicial subsets  $A \supset B$  of  $Y$  we shall denote by  $C(A, B)$  the (simplicial) chain  $Q_{n-1}$ -complex  $Q_{n-1} \otimes_{\Lambda_{n-1}} C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$ . Clearly

$$H_i(A, B; Q_{n-1}) = H_i(C(A, B)) = Q_{n-1} \otimes_{\Lambda_{n-1}} H_i(A, B; \Lambda_{n-1}).$$

Consider the short exact sequence of chain  $Q_{n-1}$ -complexes

$$(5) \quad 0 \rightarrow C(X) \rightarrow C(Y) \rightarrow C(Y, X) \rightarrow 0.$$

Provide the homology modules of complexes  $C(X)$ ,  $C(Y)$ ,  $C(Y, X)$  with bases as follows. It is evident that  $H_i(C(Y, X)) = 0$  for  $i \neq 2, m$  and

$$H_i(C(Y, X)) = H_i(C(U, \partial U)) = H_i(U, \partial U; Q_{n-1}) = Q_{n-1}$$

for  $i = 2, m$ . Fix a lifting  $\tilde{U} \subset \tilde{Y}$  of  $U \approx S^{m-2} \times D^2$ . Fix in  $H_m(C(Y, X))$  the generator  $[\tilde{U}, \partial \tilde{U}]$ . Fix in  $H_2(C(Y, X))$  the generator  $[\Delta, \partial \Delta]$  where  $\Delta$  is the meridional disk of  $\tilde{U}$ .

It follows from Claim 1 that  $H_i(C(X)) = H_i(C(Y)) = 0$  for  $i \neq 1, m - 1$ . Fix an arbitrary basis  $f$  in the  $(n-2)$ -dimensional vector  $Q_{n-1}$ -space  $H_1(Y; Q_{n-1})$ . Fix the dual basis  $g$  in  $H_{m-1}(Y; Q_{n-1})$ . It follows from Claim 2 that inclusion homomorphisms  $H_i(C(X)) \rightarrow H_i(C(Y))$  are surjective for all  $i$ . Let  $F$  and  $G$  be sequences of  $n - 2$  vectors in  $H_1(C(X))$  and in  $H_{m-1}(C(X))$  whose images under these inclusion homomorphisms are equal respectively to  $f$  and  $g$ . Claim 2 implies that  $[\partial \tilde{U}], G$  is a basis in  $H_{m-1}(C(X))$  and



$[\partial\Delta]$ ,  $F$  is a basis in  $H_1(C(X))$ . Now all homology modules of complexes  $C(X)$ ,  $C(Y)$ ,  $C(Y, X)$  are provided with bases.

Provide the modules of  $C(X)$ ,  $C(Y)$ ,  $C(Y, X)$  with natural bases (see Sec. 2.2). We may choose these bases to be compatible in the sense of Lemma 2.1.1. According to this Lemma

$$\tau(C(Y)) = \pm \tau(C(X))\tau(C(Y, X))\tau(\mathcal{H})$$

where  $\mathcal{H}$  is the homology sequence associated with the exact sequence (5). It is evident that  $\tau(\mathcal{H}) = \pm 1$ . It is easy to verify that  $\tau(C(Y, X)) = \tau(C(U, \partial U)) = \pm 1$ . (Indeed, the pair  $(U, \partial U)$  has a cell structure such that  $\text{Int } U$  contains 2 open cells; the meridional disc and its complement; for such cell structure the equality  $\tau(C(U, \partial U)) = \pm 1$  is evident. The case of an arbitrary cell structure (or triangulation) follows from the invariance of torsion under cell subdivision). Thus  $\tau(C(Y)) = \pm \tau(C(X))$ . Note that  $\tau(C(Y))$  represents  $\omega(Y)$ . Therefore  $\tau(C(X))$  also represents  $\omega(Y)$ .

Consider the chain complex

$$C = Q_n^0 \otimes_{\Lambda_n} C_*(\tilde{X}; \mathbf{Z}).$$

Note that  $Q_n^0$  is a local ring with the maximal ideal  $(t_n - 1)Q_n^0$  and associated field  $Q_{n-1}$ . Clearly,  $Q_{n-1} \otimes_{Q_n^0} C = C(X)$ . The natural bases in chain modules of  $C(X)$  lift to natural bases in chain modules of  $C$ . Claim 1 implies that for all  $i \geq 0$

$$\text{rk}_{Q_n^0} H_i(C) = \text{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \text{rk}_{Q_{n-1}} H_i(C(X)).$$

Therefore we may apply Lemma 3.4 to complexes  $C$ ,  $C(X)$ . This lemma shows that:  $H_i(C) = H_i(C(X)) = 0$  for  $i \neq 1, m - 1$ ; the basis  $[\partial\Delta]$ ,  $F$  in  $H_1(C(X))$  lifts to a basis, say,  $f_0, f_1, \dots, f_{n-2}$  in  $H_1(C)$ ; the basis  $[\partial\tilde{U}]$ ,  $G$  in  $H_{m-1}(C(X))$  lifts to a basis, say,  $g_0, g_1, \dots, g_{n-2}$  in  $H_{m-1}(C)$ ; the submodules of cycles and boundaries of  $C$  are free in all dimensions. Thus we may apply the constructions of Sec. 2.1 to  $C$  which gives rise to a torsion  $\tau(C) \in Q_n^0$ . It follows directly from the formula (3) that  $\phi(\tau(C)) = \tau(C(X))$ . Thus  $\phi(\tau(C))$  represents  $\omega(Y)$ .

Let  $v$  be the matrix of the semi-linear intersection pairing

$$\langle \ , \ \rangle : H_1(X; Q_n^0) \times H_{m-1}(X; Q_n^0) \rightarrow Q_n^0$$

with respect to bases  $f_0, f_1, \dots, f_{n-2}$  and  $g_0, g_1, \dots, g_{n-2}$ . (Here  $H_i(X; Q_n^0) = H_i(C)$ ). It is clear that  $\tau(C) (\det v)^{-1}$  represents  $\omega(X)$ . Put  $\omega = \tau(C) (\det v)^{-1}$ . We shall prove that

$$(6) \quad \det v = \pm (t_n - 1) + (t_n - 1)^2 a$$

where  $a \in Q_n^0$ . Then  $(t_n - 1)\omega \in Q_n^0$  and

$$\varphi((t_n - 1)\omega) = \varphi(\tau(C)[\pm 1 + (t_n - 1)a]^{-1}) = \pm \varphi(\tau(C)) \in \omega(Y).$$

This would complete the proof of (4.1.2).

It is obvious that

$$v = \begin{bmatrix} \langle f_0, g_0 \rangle & (t_n - 1)\alpha \\ (t_n - 1)\beta & E + (t_n - 1)\gamma \end{bmatrix}$$

where  $\alpha, \beta, \gamma$  are respectively a  $(n-2)$ -row,  $(n-2)$ -column and  $(n-2) \times (n-2)$ -matrix over  $Q_n^0$ . It turns out that

$$(7) \quad \langle f_0, g_0 \rangle = \pm (t_n - 1) + (t_n - 1)^2 b$$

with  $b \in Q_n^0$ . This immediately implies (6).

I shall prove (7) for a special choice of  $f_0$  which is sufficient for our aims. Let  $\theta: [0, 1] \rightarrow \partial\tilde{X}$  be a path whose projection to  $\tilde{Y}$  is a loop parametrizing  $\partial\Delta \subset \partial\tilde{U}$ . Let  $\eta: [0, 1] \rightarrow \tilde{X}$  be a path such that  $\eta(0) = \theta(0)$  and  $\eta(1) = t_1 \cdot \theta(0)$ . Consider the singular chain  $\mathfrak{g} = \theta - t_1\theta + t_n\eta - \eta$ . It is easy to check up that  $\mathfrak{g}$  is a cycle in  $\tilde{X}$  and that its homology class  $[\mathfrak{g}] \in H_1(C)$  projects to  $(1 - t_1)[\partial\Delta] \in H_1(C(X))$ . Put  $f_0 = (1 - t_1)^{-1}[\mathfrak{g}]$ . Then  $\langle f_0, g_0 \rangle = (1 - t_1)^{-1}\langle [\mathfrak{g}], g_0 \rangle = (1 - t_1)^{-1}(t_n - 1)\langle \eta, g_0 \rangle$  where in the right part the brackets  $\langle \cdot, \cdot \rangle$  denote the intersection pairing

$$H_1(X, \partial X; Q_n^0) \times H_{m-1}(X; Q_n^0) \rightarrow Q_n^0.$$

The image of  $\langle \eta, g_0 \rangle$  under  $\varphi: Q_n^0 \rightarrow Q_{n-1}$  can be computed using the analogous pairing

$$H_1(X, \partial X; Q_{n-1}) \times H_{m-1}(X; Q_{n-1}) \rightarrow Q_{n-1}.$$

Namely,  $\varphi(\langle \eta, g_0 \rangle) = \pm (t_1 - 1)$ . Thus  $\langle \eta, g_0 \rangle = \pm (t_1 - 1) + (t_n - 1)c$  with  $c \in Q_n^0$ . Therefore  $\langle f_0, g_0 \rangle = \pm (t_n - 1) + (t_n - 1)^2 b$  where  $b = (1 - t_1)^{-1}c$ . This implies (7).

**4.2. Proof of Theorem 2.** We may assume that  $\Delta_{u-1}(L) \neq 0$  and  $l_1 = l_2 = \dots = l_{n-1} = 0$ . Then the  $n$ -th component of  $K$  lifts to the maximal abelian covering of the exterior  $Y$  of  $L$ . The remaining part of the proof is analogous to the proof of Theorem 1. Note, however, the necessary changes. In Claim 1 for  $i = 1, 2$

$$\text{rk}_{\Lambda_n} H_i(X; \Delta_n) = \text{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = u - 1; \text{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = u - 2.$$

In the proof of (4.1.1) one should take into account that  $\text{Tors}_{\Lambda_{n-1}} H_1(X; \Lambda_{n-1})$  injects into  $\text{Tors}_{\Lambda_{n-1}} H_1(Y; \Lambda_{n-1})$  and thus the order of the first of these 2 modules divides the order of the second one.

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