

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 34 (1988)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON TORRES-TYPE RELATIONS FOR THE ALEXANDER
POLYNOMIALS OF LINKS
Autor: Turaev, V. G.
DOI: <https://doi.org/10.5169/seals-56589>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 28.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ON TORRES-TYPE RELATIONS FOR THE ALEXANDER POLYNOMIALS OF LINKS

by V. G. TURAEV

§ 1. INTRODUCTION

The classical formula of Torres [5] relates the (first) Alexander polynomial of a link K in S^3 with that of the sublink of K obtained by deleting a component. The aim of the present paper is to establish a Torres-type formula for Alexander polynomials of higher-dimensional links. We also discuss analogous formulas for higher Alexander polynomials of links in S^3 .

An n -component link in the sphere S^m is an ordered collection of n disjoint smooth imbedded oriented $(m-2)$ -dimensional spheres in S^m . With each odd-dimensional link $K \subset S^{2r+1}$ one associates a Λ_n -module $H_r(\tilde{X})$, where Λ_n is the Laurent polynomial ring $\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, X is the exterior of K and \tilde{X} is the maximal abelian covering of X . The module $H_r(\tilde{X})$ algebraically gives rise to a sequence of Fitting (or determinantal) invariants $\Delta_1(K), \Delta_2(K), \dots$, which are elements of Λ_n defined up to multiplication by monomials $\pm t_1^{s_1} \dots t_n^{s_n}$ (see [1] or § 3). The polynomial $\Delta_i(K)$ is called the i -th Alexander polynomial of K . The first Alexander polynomial $\Delta_1(K)$ is also denoted by $\Delta(K)$ and called "the Alexander polynomial of K ".

THEOREM (Torres [5]). *Let K be an n -component link in S^3 with $n \geq 2$ and let L be the sublink of K obtained by deleting the n -th component. Then*

$$\Delta(K)(t_1, \dots, t_{n-1}, 1) = \begin{cases} (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1)\Delta(L) & \text{if } n > 2 \\ \frac{t_1^{l_1} - 1}{t_1 - 1} \Delta(L) & \text{if } n = 2 \end{cases}$$

where l_i denotes the linking number of the i -th and n -th components of K .

The following theorem can be considered as a high-dimensional variant of the Torres theorem.

THEOREM 1. Let K be an n -component link in S^m with odd $m \geq 5$. Let L be the sublink of K obtained by deleting the n -th component. Then there exists an element λ of Λ_{n-1} such that

$$(1) \quad \Delta(L) = \Delta(K)(t_1, \dots, t_{n-1}, 1) \cdot \lambda \bar{\lambda}.$$

Here the overbar denotes the involution of the Laurent polynomial ring Λ_{n-1} which sends each polynomial $f(t_1, \dots, t_{n-1})$ into $f(t_1^{-1}, \dots, t_{n-1}^{-1})$.

It is well known that for any link $K \subset S^m$ with odd $m \geq 5$ the Alexander polynomial $\Delta(K)$ is non-zero. Moreover,

$$\text{aug}(\Delta(K)) = \Delta(K)(1, 1, \dots, 1) = \pm 1$$

(see [1]). This implies that $\text{aug}(\lambda) = \pm 1$ for any λ satisfying (1). It seems that there are no other restrictions on λ ; one may even guess that for any $\Delta \in \Lambda_n$, $\lambda \in \Lambda_{n-1}$ with $\text{aug}(\Delta) = \text{aug}(\lambda) = \pm 1$ and $\bar{\Delta} \doteq \Delta$ there exists a pair K, L as in Theorem 1 such that $\Delta(K) \doteq \Delta$ and $\Delta(L) \doteq \Delta(t_1, \dots, t_{n-1}, 1)\lambda\bar{\lambda}$. Here and below the symbol \doteq denotes the equality of Laurent polynomials up to multiplication by a monomial $\pm t_1^{s_1} \dots t_n^{s_n}$.

Let us call two Laurent polynomials $\Delta, \Delta' \in \Lambda_n$ algebraically cobordant if there exist polynomials $\lambda, \lambda' \in \Lambda_n$ such that $\Delta\lambda\bar{\lambda} \doteq \Delta'\lambda'\bar{\lambda}'$ and $\text{aug}(\lambda) = \text{aug}(\lambda') = \pm 1$. This terminology is suggested by the fact that Alexander polynomials of (smoothly) cobordant links are algebraically cobordant (see [4]). The formula (1) enables us to calculate Alexander polynomials of all sublinks of a given link, up to algebraic cobordism. It is curious to note that if K, K' are n -component links in S^m with odd $m \geq 5$ and if polynomials $\Delta(K), \Delta(K')$ are algebraically cobordant then Theorem 1 implies that Alexander polynomials of corresponding sublinks of K, K' are algebraically cobordant to each other. This fact reflects the evident property of geometric cobordisms: corresponding sublinks of cobordant links are cobordant.

I do not know if it is possible to associate with a link K some preferred $\lambda = \lambda(K)$ satisfying (1).

The remaining part of the Introduction is concerned with the classical links. The symbols $K, L, n, l_1, \dots, l_{n-1}$ denote the same objects as in the Torres theorem formulated above. It may well happen that some of the Alexander polynomials $\Delta_1(K), \Delta_2(K), \dots$ are equal to zero. Denote by $u = u(K)$ the minimal integer $u \geq 1$ such that $\Delta_u(K) \neq 0$. Since $\Delta_{i+1}(K)$ divides $\Delta_i(K)$ for all i , $\Delta_i(K) = 0$ for $i < u$ and $\Delta_i(K) \neq 0$ for $i \geq u(K)$.

In view of the Torres theorem it is natural to look for a relationship between $\Delta_{u(K)}(K)$ and a corresponding invariant of L . In the case $u(K) = 1$ we have the Torres formula, so we shall restrict ourselves to the case $u(K) \geq 2$ (i.e. the case $\Delta(K) = 0$).

The integers $u(K), u(L)$ are related by the inequality $u(L) \geq u(K) - 1$ (see [1] or § 4). If $l_i \neq 0$ at least for one $i = 1, \dots, n - 1$ then the stronger inequality holds: $u(L) \geq u(K)$. These inequalities suggest to relate $\Delta_u(K)$ (where we put $u = u(K)$) with $\Delta_{u-1}(L)$ and $\Delta_u(L)$. The following relationship between $\Delta_u(K)$ and $\Delta_u(L)$ was established in [4].

THEOREM ([4, Theorem 5.5.1]). *If $u = u(K) \geq 2$ then there exist an element λ of Λ_{n-1} and a subset β of the set $\{1, 2, \dots, n-1\}$ such that*

$$(2) \quad (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1) \Delta_u(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$

Several remarks are in order. a) The non-trivial case of the Theorem is the case where at least one of the integers l_1, \dots, l_{n-1} is non-zero: otherwise $t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1 = 0$ and we may put $\lambda = 0$. b) Formula (2) is proved in [4] under the additional condition $u(L) = u(K)$. However if $u(L) < u(K)$ then we have the trivial case $l_1 = l_2 = \dots = l_{n-1} = 0$; if $u(L) > u(K)$ then $\Delta_{u(K)}(L) = 0$ and we may put $\lambda = 0$. c) Formula (2) combines the factors from the Torres formula, formula (1) and a new factor $\prod (t_i - 1)$. All these factors may be non-trivial (see [4]). d) An explicit construction of the set $\beta = \beta(K)$ is given in [4, § 5]. I do not know if there exists a preferred $\lambda = \lambda(K)$ which satisfies (2).

The relationships between the polynomials $\Delta_u(K)$ and $\Delta_{u-1}(L)$ were first considered by Levine [2] in the case $u = 2$.

THEOREM (Levine [2]). *If $u(K) \geq 2$ then there exist an element $\lambda \in \Lambda_{n-1}$ and a set $\beta \subset \{1, 2, \dots, n-1\}$ such that*

$$\Delta(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_2(K) (t_1, \dots, t_{n-1}, 1).$$

Note that in the case $u(K) > 2$ the Levine's theorem is evident: if $u(K) > 2$ then $u(L) \geq u(K) - 1 > 1$ so that $\Delta(L) = \Delta_2(K) = 0$.

The following theorem generalizes the Levine's result.

THEOREM 2. *If $u = u(K) \geq 2$ then there exist an element λ of Λ_{n-1} and a set $\beta \subset \{1, 2, \dots, n-1\}$ such that*

$$\Delta_{u-1}(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$

The non-trivial case of Theorem 2 is the case $l_1 = l_2 = \dots = l_{n-1} = 0$: otherwise $u(L) \geq u$ so that $\Delta_{u-1}(L) = 0$ and we may put $\lambda = 0$.

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

This research was completed while the author was visiting the University of Geneva. I thank the staff of the Mathematical Department of the University and especially professors J.-C. Hausmann and M. Kervaire for their hospitality.

§ 2. TORSIONS OF CHAIN COMPLEXES AND MANIFOLDS

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let Q be a field. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are two bases of a Q -module then $a_i = \sum_{j=1}^n c_{i,j} b_j$ where $(c_{i,j})$ is a non-singular $n \times n$ -matrix over Q ; the determinant $\det(c_{i,j}) \in Q \setminus 0$ is denoted by $[a/b]$.

Let $C = (C_m \rightarrow \dots \rightarrow C_0)$ be a chain Q -complex. Suppose that each Q -module C_i is finite dimensional with a preferred basis c_i and each Q -module $H_i(C)$ also has a preferred basis h_i . (The case $C_i = 0$ or $H_i(C) = 0$ is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion $\tau(C) \in Q$ as follows. For each $i = 1, 2, \dots, m$ choose a sequence $b_i = (b_1^i, \dots, b_{r_i}^i)$ of elements of C_i such that $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$ is a basis in $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$. For each $i = 0, 1, \dots, m$ choose a lifting \tilde{h}_i of the basis h_i to $\text{Ker } \partial_{i-1}$. The combined sequence $\partial_i(b_{i+1})\tilde{h}_i b_i$ is a basis in C_i . (It is understood that $b_0 = \emptyset$ and $b_{m+1} = \emptyset$). Put

$$(3) \quad \tau(C) = \prod_{i=0}^m [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where $\varepsilon(i) = (-1)^{i+1}$. Clearly, $\tau(C) \in Q \setminus 0$. It is easy to verify that $\tau(C)$ does not depend on the choice of b_i and \tilde{h}_i .

(Note that the torsion of C defined in Milnor's survey article [3] equals $\pm \tau(C)^{-1} \in Q / \pm 1$ and that Milnor uses the additive notation for the multiplication in $Q \setminus 0 = K_1(Q)$.)

2.1.1. LEMMA (multiplicativity of torsion). *Let $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ be a short exact sequence of m -dimensional chain complexes over a field Q .*

Suppose that for all $i = 0, 1, \dots, m$ the modules C_i, C'_i, C''_i are provided with preferred bases c'_i, c_i, c''_i which are compatible, in the sense that $[c'_i c''_i / c_i] = \pm 1$. Suppose that for all $i = 0, 1, \dots, m$ the homology modules $H_i(C), H_i(C'), H_i(C'')$ are provided with preferred bases. Let \mathcal{H} be the homology sequence of the sequence $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$:

$$\mathcal{H} = (H_m(C') \rightarrow H_m(C) \rightarrow \dots \rightarrow H_0(C) \rightarrow H_0(C'')).$$

Consider \mathcal{H} as an acyclic based chain complex over Q . Then $\tau(C) = \pm \tau(C')\tau(C'')\tau(\mathcal{H})$.

For a proof see [3].

2.2. THE TORSION ω . Let M be an orientable compact smooth manifold of odd dimension m with $\text{rg } H_1(M) \geq 1$. Denote the free abelian group $H_1(M)/\text{Tors } H_1(M)$ by G . Denote the fraction field of the group ring $\mathbf{Z}[G]$ by Q . Provide Q with the involution $q \mapsto \bar{q}$ which sends $g \in G$ to g^{-1} . The field Q defines via the natural homomorphism $\mathbf{Z}[\pi_1(M)] \rightarrow Q$ a system of local coefficients on M . We shall denote this system by the same symbol Q . Assume that $H_*(\partial M; Q) = 0$. In this setting one can consider a torsion-type invariant $\omega(M)$ of M which is "an element of $Q \setminus 0$ defined up to multiplication by $\pm gq\bar{q}$ with $g \in G$ and $q \in Q \setminus 0$ " (see [4]).

Recall the definition of $\omega(M)$ given in [4, § 5]. Let $\tilde{M} \rightarrow M$ be the regular covering of M corresponding to the kernel of the natural homomorphism $\pi_1(M) \rightarrow G$. Fix a C^1 -triangulation of M and the induced G -equivariant triangulation of \tilde{M} . Choose over each simplex of the (fixed) triangulation of M a simplex of the triangulation of \tilde{M} . These simplices in \tilde{M} being arbitrarily oriented and ordered determine "natural" bases of the modules of the simplicial chain $\mathbf{Z}[G]$ -complex $C_*(\tilde{M}; \mathbf{Z})$. These bases induce "natural" Q -bases in the chain Q -complex

$$C = Q \otimes_{\mathbf{Z}[G]} C_*(\tilde{M}; \mathbf{Z}).$$

For all $i = 0, 1, \dots, m$ choose an arbitrary Q -basis h_i in $H_i(M; Q) = H_i(C)$. Denote by $\tau(C, h_0, \dots, h_m)$ the torsion of C with respect to the bases in chain modules constructed above and the bases h_0, h_1, \dots, h_m in homology. Since $H_*(\partial M; Q) = 0$ the semi-linear intersection form $H_i(M; Q) \times H_{m-i}(M; Q) \rightarrow Q$ is non-singular. Let v_i be the matrix of this form regarding the bases h_i and h_{m-i} . Put

$$d = \tau(C, h_0, h_1, \dots, h_m) \prod_{i=0}^r (\det v_i)^{-\varepsilon(i)} \in Q \setminus 0$$

where $r = (m-1)/2$ and $\varepsilon(i) = (-1)^{i+1}$. It is easy to show that under a different choice of natural bases and bases h_0, h_1, \dots, h_m the element d is replaced by $\pm gq\bar{q}d$ with $g \in G, q \in Q \setminus 0$. Thus the set $\{\pm gq\bar{q}d \mid g \in Q \setminus 0\} \subset Q$ does not depend on the choice of bases. It also does not depend on the choice of triangulation in M . It is this set which is $\omega(M)$.

An explicit formula established in [4] enables us to calculate $\omega(M)$ in terms of the orders of $\mathbf{Z}[G]$ -modules $H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; \mathbf{Z}), H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$ and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by J the image of the inclusion homomorphism $H_r(\partial\tilde{M}) \rightarrow H_r(\tilde{M})$ where $r = (m-1)/2$. Then up to multiples of type $q\bar{q}$ with $q \in Q \setminus 0$

$$(4) \quad \omega(M) = \text{ord}(\text{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)) (\text{ord } J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\text{ord } H_i(\partial M)]^{\varepsilon(i)}$$

(see [4, Theorem 5.1.1]). Note that the equalities $Q \otimes_{\mathbf{Z}[G]} H_*(\partial\tilde{M}) = H_*(\partial\tilde{M}; Q) = 0$ imply that $H_*(\partial\tilde{M})$ and J are torsion $\mathbf{Z}[G]$ -modules. Therefore $\text{ord } H_i(\partial\tilde{M})$ and $\text{ord } J$ are non-zero elements of $\mathbf{Z}[G]$.

We shall apply formula (4) in the case where M is the exterior of an n -component link $K \subset S^m$ with odd m . The condition $H_*(\partial M; Q) = 0$ is always fulfilled in this case. Here the field Q is canonically identified with the field of rational functions of n variables $Q_n = Q(t_1, \dots, t_n)$. Thus $\omega(M) \subset Q_n$. If $m \geq 5$ then (4) implies that

$$\Delta(K)(t_1, \dots, t_n) \cdot \prod_{i=1}^n (t_i - 1) \subset \omega(M).$$

If $m = 3$ then there exists a unique subset $\alpha = \alpha(K)$ of the set $\{1, 2, \dots, n\}$ such that

$$\Delta_{u(K)}(K)(t_1, \dots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M).$$

For proofs and details consult [4, § 5].

§ 3. ALGEBRAIC LEMMAS

3.1. PRELIMINARY DEFINITIONS. For a finitely generated module H over a (commutative) domain R we denote by $\text{rk}_R H$ or, briefly, by $\text{rk } H$ the integer $\dim_Q(Q \otimes_R H)$ where $Q = Q(R)$ denotes the field of fractions of R . For a R -linear homomorphism $f: H \rightarrow H'$ we put $\text{rk } f = \text{rk}_R f(H)$. Note that if \bar{R} is the localization of R at some multiplicative system then $Q(\bar{R}) = Q(R)$ and therefore the (exact) functor $(H \mapsto \bar{R} \otimes_R H, f \mapsto \text{id}_{\bar{R}} \otimes f)$

preserves the ranks of modules and homomorphisms. If H, H' are finitely generated free R -modules and if A is the matrix of a R -homomorphism $H \rightarrow H'$ with respect to some bases then $\text{rk } f = \text{rk } A$ where $\text{rk } A$ is the maximal integer r such that some $r \times r$ -minor of A is non-zero.

If R is a unique factorization domain with 1 and if A is a matrix with $n < \infty$ columns and possibly infinite number of rows then $\Delta_i(A)$ denotes the greatest common divisor of the $(n-i+1) \times (n-i+1)$ -minors of A . Here $i = 1, 2, \dots$ and $\Delta_i(A)$ is an element of R defined up to a unit multiple. If H is a finitely generated module over R and A is a presentation matrix of H then $\Delta_i(A)$ depends only on H and i ; one defines $\Delta_i(H) = \Delta_i(A)$. Clearly $\Delta_i(H) = 0$ for $i \leq \text{rg } H = n - \text{rg } A$ and $\Delta_i(H) \neq 0$ for $i > \text{rg } H$. The invariant $\Delta_1(H)$ is denoted also by $\text{ord } H$; it is called the order of H . It is clear that $\text{ord } H \neq 0$ iff $H = \text{Tors}_R H$. For proofs and further information see [1].

Recall, finally, that a local ring is a domain K which has a unique maximal (proper) ideal. The quotient of K by this ideal is a field which we shall call "the field associated to K ".

3.2. LEMMA. *Let R, R' be (commutative) domains with 1 and let $\varphi: R \rightarrow R'$ be a ring homomorphism. Let $C = (\dots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \dots)$ be a finitely generated free chain complex over R and let C' be the chain R' -complex $R' \otimes_R C$. Then: (i) $\text{rk}_{R'} H_i(C') \geq \text{rk}_R H_i(C)$ and $\text{rk } \partial'_i \leq \text{rk } \partial_i$ for all i where ∂_i, ∂'_i are the boundary homomorphisms $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$; (ii) if $\text{rk } H_i(C') = \text{rk } H_i(C)$ for some i then $\text{rk } \partial'_j = \text{rk } \partial_j$ for $j = i, i+1$; (iii) if R, R' are unique factorization Noetherian domains and if $\text{rk } H_i(C') = \text{rk } H_i(C)$ then $\varphi(\text{ord}(\text{Tors}_R H_i(C)))$ divides $\text{ord}(\text{Tors}_{R'} H_i(C'))$.*

Proof. Let $n = \text{rk } C_i$. Let $A = (a_{p,q}), 1 \leq q \leq n, 1 \leq p$, be the matrix of ∂_i with respect to some bases in C_i, C_{i+1} . Then $A' = (\varphi(a_{p,q}))$ is the matrix of ∂'_i with respect to the induced bases in C'_i, C'_{i+1} . It is evident that $\text{rk } \partial'_i = \text{rk } A' \leq \text{rk } A = \text{rk } \partial_i$. Therefore

$$\text{rk } H_i(C') = n - \text{rk } \partial'_i - \text{rk } \partial'_{i+1} \geq n - \text{rk } \partial_i - \text{rk } \partial_{i+1} = \text{rk } H_i(C).$$

These inequalities imply (i) and (ii).

Put $r = n - \text{rk } A + 1$ and denote the R -module $C_i/\text{Im } \partial_i$ by J . Since A is a presentation matrix of J we have $\text{ord}(\text{Tors}_R J) = \Delta_r(A)$ (see [1, p. 31]). From the exact sequence $0 \rightarrow H_i(C) \rightarrow J \rightarrow C_{i-1}$ we obtain that $\text{Tors } J = \text{Tors } H_i(C)$. Thus $\text{ord}(\text{Tors } H_i(C)) = \Delta_r(A)$. Analogously $\text{ord}(\text{Tors } H_i(C')) = \Delta_{r'}(A')$ where $r' = n - \text{rk } A' + 1$. If $\text{rk } H_i(C) = \text{rk } H_i(C')$ then $\text{rk } A = \text{rk } A'$ and therefore $r = r'$. It is evident that $\varphi(\Delta_j(A))$ divides $\Delta_j(A')$ for all j . This implies (iii).

3.3. LEMMA. Let R be a local ring and F be the associated field. Let $f: C_1 \rightarrow C_0$ be a R -homomorphism of finitely generated free R -modules and let $\bar{f}: F \otimes_R C_1 \rightarrow F \otimes_R C_0$ be the induced F -homomorphism. If $\text{rk } f = \text{rk } \bar{f}$ then with respect to some bases in C_1, C_0 the homomorphism f is presented by the matrix $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where E is the unit matrix of order $\text{rk } f$.

Proof. Since F is a field we can choose bases d_0, d_1 respectively in $F \otimes_R C_0, F \otimes_R C_1$ so that the matrix of \bar{f} regarding these bases has the form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Let \mathcal{D}_i be a lifting of d_i to $C_i, i = 1, 2$. Here \mathcal{D}_i is a sequence of $\text{rg } C_i$ elements of C_i . In view of Nakayama's lemma \mathcal{D}_i generate C_i . This implies that \mathcal{D}_i generates the $(\text{rg } C_i)$ -dimensional vector space $Q(R) \otimes_R C_i$ over the field $Q(R)$. Therefore, the elements of the sequence \mathcal{D}_i are linearly independent over $Q(R)$ and, hence, over R . Thus \mathcal{D}_i is a basis of C_i for $i = 0, 1$. The matrix of f with respect to bases $\mathcal{D}_0, \mathcal{D}_1$ has the form $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$ where U, X, Y, Z are matrices over the maximal ideal u of R . Note that $\det(E+U) = 1 \pmod{u}$. Since all elements of $R \setminus u$ are invertible in R the square matrix $E+U$ is invertible over R . Therefore we can choose bases in C_0, C_1 so that the corresponding matrix of f equals $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$. Since $\text{rk } f = \text{rk } \bar{f} = \text{rk } E, Y' = 0$.

3.4. LEMMA. Let R be a local ring and F be the associated field. Let $C = (\dots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \dots)$ be a finitely generated free chain complex over R . Let C' be the chain F -complex $F \otimes_R C$. Let ∂_i, ∂'_i be the boundary homomorphisms $C_{i+1} \rightarrow C_i, C'_{i+1} \rightarrow C'_i$. If $\text{rk}_R H_i(C) = \text{rk}_F H_i(C')$ for some i then: $H_i(C), \text{Im } \partial_{i+1}, \text{Im } \partial_i$ are free R -modules and $C_i = \text{Im } \partial_{i+1} \oplus H_i(C) \oplus \text{Im } \partial_i$; the projection $C \rightarrow C'$ induces F -isomorphisms $F \otimes_R H_i(C) \rightarrow H_i(C'), F \otimes_R \text{Im } \partial_j \rightarrow \text{Im } \partial'_j$ with $j = i, i+1$.

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

§ 4. PROOF OF THEOREMS 1 AND 2

4.1. PROOF OF THEOREM 1. Denote by Q_n the fraction field of the ring $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. Denote by Q_n^0 the subring of Q_n which consists of rational functions fg^{-1} with $f, g \in \Lambda_n$ and $g \notin (t_n - 1)\Lambda_n$ (so that

$g(t_1, \dots, t_{n-1}, 1) \neq 0$). The homomorphism $f \mapsto f(t_1, \dots, t_{n-1}, 1): \Lambda_n \rightarrow \Lambda_{n-1}$ uniquely extends to a ring homomorphism $Q_n^0 \rightarrow Q_{n-1}$ which is denoted by φ .

Denote by X the exterior of K and by Y the exterior of L .

We shall prove the following two statements.

$$(4.1.1). \quad \varphi(\Delta(K)) = \Delta(K)(t_1, \dots, t_{n-1}, 1) \text{ divides } \Delta(L) \text{ in } \Lambda_{n-1}.$$

(4.1.2). There exists a representative ω of the torsion $\omega(X) \subset Q_n$ such that $(t_n - 1)\omega \in Q_n^0$ and $\varphi((t_n - 1)\omega)$ represents $\omega(Y) \subset Q_{n-1}$.

Let us show first that these two statements imply the Theorem. Let ω be the element of Q_n produced by (4.1.2). Put $\pi = \prod_{i=1}^{n-1} (t_i - 1)$. According to the results formulated in Sec. 2.2 the product $(t_n - 1)\pi \cdot \Delta(K)$ represents $\omega(X)$. Thus

$$\omega \doteq \frac{f\bar{f}}{g\bar{g}}(t_n - 1)\pi\Delta(K)$$

where $f, g \in \Lambda_n \setminus 0$. We may assume that $f\bar{f}$ and $g\bar{g}$ are relatively prime. If $t_n - 1$ does not divide g then $\omega \in Q_n^0$ and $\varphi((t_n - 1)\omega) = 0$ which contradicts to the inclusion $\varphi((t_n - 1)\omega) \in \omega(Y)$. Thus $g = (t_n - 1)h$ with $h \in \Lambda_n$. In view of (4.1.1), $\varphi(\Delta(K)) \neq 0$, i.e. $t_n - 1$ does not divide $\Delta(K)$. If $\varphi(h) = 0$ then $(t_n - 1)^2$ divides g which obviously contradicts the inclusion $(t_n - 1)\omega \in Q_n^0$. Thus $\varphi(h) \neq 0$. We have

$$h\bar{h}(t_n - 1)\omega \doteq f\bar{f}\pi\Delta(K).$$

Since $\varphi(h\bar{h}(t_n - 1)\omega) \neq 0$ we have $\varphi(f) \neq 0$. This implies that $\pi \cdot \varphi(\Delta(K)) \doteq q\bar{q}\varphi((t_n - 1)\omega)$ where $q = \varphi(h)/\varphi(f)$. Thus $\pi\varphi(\Delta(K))$ represents $\omega(Y)$. Since $\pi\Delta(L) \in \omega(Y)$ we have

$$\varphi(\Delta(K))\lambda\bar{\lambda} = \Delta(L)\mu\bar{\mu}$$

with non-zero $\lambda, \mu \in \Lambda_{n-1}$. We may assume that $\lambda\bar{\lambda}$ and $\mu\bar{\mu}$ are relatively prime. Since $\varphi(\Delta(K))$ divides $\Delta(L)$ we immediately obtain $\mu\bar{\mu} = 1$. Thus, $\Delta(L) = \varphi(\Delta(K))\lambda\bar{\lambda}$.

Let us prove (4.1.1) and (4.1.2). We may assume that $X \subset Y$ and that $Y \setminus X$ is the interior of the regular neighborhood $U \subset Y$ of the n -th component of K in Y . Let $p: \tilde{X} \rightarrow X$ and $q: \tilde{Y} \rightarrow Y$ be the maximal abelian coverings with the groups of covering transformations respectively $H_1(X) \approx \mathbf{Z}^n$ (generators t_1, \dots, t_n) and $H_1(Y) \approx \mathbf{Z}^{n-1}$ (generators t_1, \dots, t_{n-1}). It is clear that p is the composition of an infinite cyclic covering $\tilde{X} \rightarrow q^{-1}(X)$ and the covering $q: q^{-1}(X) \rightarrow X$.

Fix a C^1 -triangulation of Y so that X and U are simplicial subcomplexes of Y . Fix also the induced equivariant triangulations in \tilde{X} and \tilde{Y} .

The ring Λ_{n-1} determines via the natural homomorphism $\mathbf{Z}[\pi_1(Y)] \rightarrow \mathbf{Z}[H_1 Y] = \Lambda_{n-1}$ a system of local coefficients on Y which we denote by the same symbol Λ_{n-1} . According to definitions, for any simplicial subsets $A \supset B$ of Y the Λ_{n-1} -module $H_*(A, B; \Lambda_{n-1})$ equals $H_*(C(q^{-1}(A), q^{-1}(B); \mathbf{Z}))$. Here the simplicial chain complex $C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$ is a finitely generated free Λ_{n-1} -complex. Analogously Λ_n defines a system of local coefficients on X and for simplicial subsets $A \supset B$ of X the Λ_n -module $H_*(A, B; \Lambda_n)$ equals $H_*(C(p^{-1}(A), p^{-1}(B); \mathbf{Z}))$. Note that

$$\Lambda_{n-1} \otimes_{\Lambda_n} C_*(p^{-1}(A), p^{-1}(B); \mathbf{Z}) = C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$$

where Λ_n acts on Λ_{n-1} via φ .

Claim 1. For $i \neq 1, m-1$,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = 0.$$

For $i = 1, m-1$,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = n-1; \quad \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = n-2.$$

Proof of Claim 1. We shall compute the rank of $H_i(X; \Lambda_n)$; modules $H_i(X; \Lambda_{n-1})$ and $H_i(Y; \Lambda_{n-1})$ can be treated similarly.

Denote by V a wedge of n circles in X such that the inclusion homomorphism $H_1(V; \mathbf{Z}) \rightarrow H_1(X; \mathbf{Z}) = \mathbf{Z}^n$ is bijective. Then $H_i(X, V, \mathbf{Z}) = 0$ for $i \leq m-2$. Therefore an application of Lemma 3.2(i) to complexes $C_*(\tilde{X}, p^{-1}(V); \mathbf{Z})$ and $C_*(X, V; \mathbf{Z})$ gives that $\mathrm{rk}_{\Lambda_n} H_i(X, V; \Lambda_n) = 0$ for $i \leq m-2$. This implies that $\mathrm{rk} H_i(X; \Lambda_n) = \mathrm{rk} H_i(V; \Lambda_n)$ for $i \leq m-3$ and that $\mathrm{rk} H_{m-2}(X; \Lambda_n) \leq \mathrm{rk} H_{m-2}(V; \Lambda_n)$. The rank of $H_i(V; \Lambda_n)$ can be computed directly: It is equal to 0 if $i \neq 1$ and to $n-1$ if $i = 1$. Thus the rank of $H_i(X; \Lambda_n)$ equals 0 if $i \neq 1, m-1$ and equals $n-1$ if $i = 1$. The equality $\mathrm{rk} H_{m-1}(X; \Lambda_n) = n-1$ follows from duality or from the equalities

$$\sum_{i=0}^m (-1)^i \mathrm{rk} H_i(X; \Lambda_n) = \chi(X) = 0.$$

Claim 2. The exact homology sequence of (Y, X) with coefficients in Λ_{n-1} splits into short exact sequences

$$\begin{aligned}
0 &\rightarrow H_m(Y, X; \Lambda_{n-1}) \rightarrow H_{m-1}(X; \Lambda_{n-1}) \rightarrow H_{m-1}(Y; \Lambda_{n-1}) \rightarrow 0, \\
0 &\rightarrow H_i(X; \Lambda_{n-1}) \xrightarrow{\cong} H_i(Y; \Lambda_{n-1}) \rightarrow 0, \quad (i \neq 1, m-1) \\
0 &\rightarrow H_2(Y, X; \Lambda_{n-1}) \xrightarrow{\partial_1} H_1(X; \Lambda_{n-1}) \rightarrow H_1(Y; \Lambda_{n-1}) \rightarrow 0.
\end{aligned}$$

Proof of Claim 2. Clearly, $H_i(Y, X; \Lambda_{n-1}) = H_i(U, \partial U; \Lambda_{n-1}) = 0$ for $i \neq 2, m$. Therefore the only thing to prove is the injectivity of ∂_1 . According to Claim 1 $\text{rk } H_1(X; \Lambda_{n-1}) = n - 1$ and $\text{rk } H_1(Y; \Lambda_{n-1}) = n - 2$. Since $H_2(Y, X; \Lambda_{n-1}) = \Lambda_{n-1}$ we see that ∂_1 is injective.

Proof of (4.1.1). In view of the equalities $\text{rg } H_i(X; \Lambda_n) = \text{rg } H_i(X; \Lambda_{n-1})$, $i = 0, 1, \dots$ we may apply Lemma 3.2 (iii) to the chain complexes $C_*(\tilde{X}; \mathbf{Z})$ and $C_*(q^{-1}(X); \mathbf{Z})$ respectively over Λ_n and Λ_{n-1} . Since $m - 1 > r > 1$ Claims 1, 2 show that $H_r(X; \Lambda_n)$ and $H_r(X; \Lambda_{n-1})$ are torsion modules respectively over Λ_n and Λ_{n-1} and $H_r(X, \Lambda_{n-1}) = H_r(Y; \Lambda_{n-1})$. By definition $\Delta(K) = \text{ord } H_r(X; \Lambda_n)$ and $\Delta(L) = \text{ord } H_r(Y; \Lambda_{n-1}) = \text{ord } H_r(X; \Lambda_{n-1})$. Lemma 3.2 (iii) directly implies that $\varphi(\Delta(K))$ divides $\Delta(L)$.

It remains to prove Statement (4.1.2) which is, of course, the core of Theorem 1. For simplicial subsets $A \supset B$ of Y we shall denote by $C(A, B)$ the (simplicial) chain Q_{n-1} -complex $Q_{n-1} \otimes_{\Lambda_{n-1}} C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$. Clearly

$$H_i(A, B; Q_{n-1}) = H_i(C(A, B)) = Q_{n-1} \otimes_{\Lambda_{n-1}} H_i(A, B; \Lambda_{n-1}).$$

Consider the short exact sequence of chain Q_{n-1} -complexes

$$(5) \quad 0 \rightarrow C(X) \rightarrow C(Y) \rightarrow C(Y, X) \rightarrow 0.$$

Provide the homology modules of complexes $C(X)$, $C(Y)$, $C(Y, X)$ with bases as follows. It is evident that $H_i(C(Y, X)) = 0$ for $i \neq 2, m$ and

$$H_i(C(Y, X)) = H_i(C(U, \partial U)) = H_i(U, \partial U; Q_{n-1}) = Q_{n-1}$$

for $i = 2, m$. Fix a lifting $\tilde{U} \subset \tilde{Y}$ of $U \approx S^{m-2} \times D^2$. Fix in $H_m(C(Y, X))$ the generator $[\tilde{U}, \partial\tilde{U}]$. Fix in $H_2(C(Y, X))$ the generator $[\Delta, \partial\Delta]$ where Δ is the meridional disk of \tilde{U} .

It follows from Claim 1 that $H_i(C(X)) = H_i(C(Y)) = 0$ for $i \neq 1, m - 1$. Fix an arbitrary basis f in the $(n-2)$ -dimensional vector Q_{n-1} -space $H_1(Y; Q_{n-1})$. Fix the dual basis g in $H_{m-1}(Y; Q_{n-1})$. It follows from Claim 2 that inclusion homomorphisms $H_i(C(X)) \rightarrow H_i(C(Y))$ are surjective for all i . Let F and G be sequences of $n - 2$ vectors in $H_1(C(X))$ and in $H_{m-1}(C(X))$ whose images under these inclusion homomorphisms are equal respectively to f and g . Claim 2 implies that $[\partial\tilde{U}], G$ is a basis in $H_{m-1}(C(X))$ and

$[\partial\Delta]$, F is a basis in $H_1(C(X))$. Now all homology modules of complexes $C(X)$, $C(Y)$, $C(Y, X)$ are provided with bases.

Provide the modules of $C(X)$, $C(Y)$, $C(Y, X)$ with natural bases (see Sec. 2.2). We may choose these bases to be compatible in the sense of Lemma 2.1.1. According to this Lemma

$$\tau(C(Y)) = \pm \tau(C(X))\tau(C(Y, X))\tau(\mathcal{H})$$

where \mathcal{H} is the homology sequence associated with the exact sequence (5). It is evident that $\tau(\mathcal{H}) = \pm 1$. It is easy to verify that $\tau(C(Y, X)) = \tau(C(U, \partial U)) = \pm 1$. (Indeed, the pair $(U, \partial U)$ has a cell structure such that $\text{Int } U$ contains 2 open cells; the meridional disc and its complement; for such cell structure the equality $\tau(C(U, \partial U)) = \pm 1$ is evident. The case of an arbitrary cell structure (or triangulation) follows from the invariance of torsion under cell subdivision). Thus $\tau(C(Y)) = \pm \tau(C(X))$. Note that $\tau(C(Y))$ represents $\omega(Y)$. Therefore $\tau(C(X))$ also represents $\omega(Y)$.

Consider the chain complex

$$C = Q_n^0 \otimes_{\Lambda_n} C_*(\tilde{X}; \mathbf{Z}).$$

Note that Q_n^0 is a local ring with the maximal ideal $(t_n - 1)Q_n^0$ and associated field Q_{n-1} . Clearly, $Q_{n-1} \otimes_{Q_n^0} C = C(X)$. The natural bases in chain modules of $C(X)$ lift to natural bases in chain modules of C . Claim 1 implies that for all $i \geq 0$

$$\text{rk}_{Q_n^0} H_i(C) = \text{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \text{rk}_{Q_{n-1}} H_i(C(X)).$$

Therefore we may apply Lemma 3.4 to complexes C , $C(X)$. This lemma shows that: $H_i(C) = H_i(C(X)) = 0$ for $i \neq 1, m - 1$; the basis $[\partial\Delta]$, F in $H_1(C(X))$ lifts to a basis, say, f_0, f_1, \dots, f_{n-2} in $H_1(C)$; the basis $[\partial\tilde{U}]$, G in $H_{m-1}(C(X))$ lifts to a basis, say, g_0, g_1, \dots, g_{n-2} in $H_{m-1}(C)$; the submodules of cycles and boundaries of C are free in all dimensions. Thus we may apply the constructions of Sec. 2.1 to C which gives rise to a torsion $\tau(C) \in Q_n^0$. It follows directly from the formula (3) that $\varphi(\tau(C)) = \tau(C(X))$. Thus $\varphi(\tau(C))$ represents $\omega(Y)$.

Let v be the matrix of the semi-linear intersection pairing

$$\langle \ , \ \rangle : H_1(X; Q_n^0) \times H_{m-1}(X; Q_n^0) \rightarrow Q_n^0$$

with respect to bases f_0, f_1, \dots, f_{n-2} and g_0, g_1, \dots, g_{n-2} . (Here $H_i(X; Q_n^0) = H_i(C)$). It is clear that $\tau(C) (\det v)^{-1}$ represents $\omega(X)$. Put $\omega = \tau(C) (\det v)^{-1}$. We shall prove that

$$(6) \quad \det v = \pm (t_n - 1) + (t_n - 1)^2 a$$

where $a \in Q_n^0$. Then $(t_n - 1)\omega \in Q_n^0$ and

$$\varphi((t_n - 1)\omega) = \varphi(\tau(C)[\pm 1 + (t_n - 1)a]^{-1}) = \pm \varphi(\tau(C)) \in \omega(Y).$$

This would complete the proof of (4.1.2).

It is obvious that

$$v = \begin{bmatrix} \langle f_0, g_0 \rangle & (t_n - 1)\alpha \\ (t_n - 1)\beta & E + (t_n - 1)\gamma \end{bmatrix}$$

where α, β, γ are respectively a $(n - 2)$ -row, $(n - 2)$ -column and $(n - 2) \times (n - 2)$ -matrix over Q_n^0 . It turns out that

$$(7) \quad \langle f_0, g_0 \rangle = \pm (t_n - 1) + (t_n - 1)^2 b$$

with $b \in Q_n^0$. This immediately implies (6).

I shall prove (7) for a special choice of f_0 which is sufficient for our aims. Let $\theta: [0, 1] \rightarrow \partial\tilde{X}$ be a path whose projection to \tilde{Y} is a loop parametrizing $\partial\Delta \subset \partial\tilde{U}$. Let $\eta: [0, 1] \rightarrow \tilde{X}$ be a path such that $\eta(0) = \theta(0)$ and $\eta(1) = t_1 \cdot \theta(0)$. Consider the singular chain $\mathfrak{g} = \theta - t_1\theta + t_n\eta - \eta$. It is easy to check up that \mathfrak{g} is a cycle in \tilde{X} and that its homology class $[\mathfrak{g}] \in H_1(C)$ projects to $(1 - t_1)[\partial\Delta] \in H_1(C(X))$. Put $f_0 = (1 - t_1)^{-1}[\mathfrak{g}]$. Then $\langle f_0, g_0 \rangle = (1 - t_1)^{-1} \langle [\mathfrak{g}], g_0 \rangle = (1 - t_1)^{-1}(t_n - 1) \langle \eta, g_0 \rangle$ where in the right part the brackets $\langle \ , \ \rangle$ denote the intersection pairing

$$H_1(X, \partial X; Q_n^0) \times H_{m-1}(X; Q_n^0) \rightarrow Q_n^0.$$

The image of $\langle \eta, g_0 \rangle$ under $\varphi: Q_n^0 \rightarrow Q_{n-1}$ can be computed using the analogous pairing

$$H_1(X, \partial X; Q_{n-1}) \times H_{m-1}(X; Q_{n-1}) \rightarrow Q_{n-1}.$$

Namely, $\varphi(\langle \eta, g_0 \rangle) = \pm (t_1 - 1)$. Thus $\langle \eta, g_0 \rangle = \pm (t_1 - 1) + (t_n - 1)c$ with $c \in Q_n^0$. Therefore $\langle f_0, g_0 \rangle = \pm (t_n - 1) + (t_n - 1)^2 b$ where $b = (1 - t_1)^{-1}c$. This implies (7).

4.2. *Proof of Theorem 2.* We may assume that $\Delta_{u-1}(L) \neq 0$ and $l_1 = l_2 = \dots = l_{n-1} = 0$. Then the n -th component of K lifts to the maximal abelian covering of the exterior Y of L . The remaining part of the proof is analogous to the proof of Theorem 1. Note, however, the necessary changes. In Claim 1 for $i = 1, 2$

$$\text{rk}_{\Lambda_n} H_i(X; \Delta_n) = \text{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = u - 1; \text{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = u - 2.$$

In the proof of (4.1.1) one should take into account that $\text{Tors}_{\Lambda_{n-1}} H_1(X; \Lambda_{n-1})$ injects into $\text{Tors}_{\Lambda_{n-1}} H_1(Y; \Lambda_{n-1})$ and thus the order of the first of these 2 modules divides the order of the second one.

REFERENCES

- [1] HILLMAN, J. A. *Alexander Ideals of Links*. Lecture Notes in Math. 895, Springer-Verlag, New York, 1981.
- [2] LEVINE, J. P. Links with Alexander Polynomial zero. *Indiana Univ. Math. J.* 36 (1987), 91-108.
- [3] MILNOR, J. W. Whitehead Torsion. *Bull. Amer. Math. Soc.* 72 (1966), 358-426.
- [4] TURAEV, V. G. Reidemeister Torsion in the Knot Theory. *Uspechi Matem. Nauk* 41 (1986), 97-147 (Russian); English translation: *Russian Math. Surveys* 41 (1986), 119-182.
- [5] TORRES, G. On the Alexander polynomial. *Annals of Math.* 57 (1953), 57-89.

(Reçu le 1^{er} juillet 1987)

V. G. Turaev

University of Geneva

Permanent address:

Leningrad Branch of Steklov Math. Inst.

Fontanka 27

Leningrad 191011 (USSR)