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Artikel: GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

Autor: Hayes, Sandra / Pourcin, Geneviève

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$$(*) \qquad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A \,.$$

For $f \in \mathcal{O}(\tilde{X})$, $a \in \tilde{X}$ and $b \in p^{-1}(p(a))$, it is now possible to conclude that $f(a) = f(b)$ is true. Let $x := \pi(a)$. Due to $(*)$, functions $g \in S_x$ and $h \in A$ exist with $f = h/g \circ \pi$. Since a and b are equivalent with respect to the equivalence relation R, $f(a) = f(b)$ follows, and a continuous function $F: Y \to \mathbf{C}$ exists with $F \circ p = f$.

Since the Stein complex structure on Y is not in general the canonical ringed quotient structure, it is still necessary to verify that F is holomorphic in order to prove the density of A in $\mathcal{O}(\tilde{X})$. To that end, let $H \in \mathcal{O}(Y)$ and $G \in \mathcal{O}(Y)$ have the property that $H \circ p = h$ and $G \circ p = g \circ \pi$. Such functions exist because $p^*(\mathcal{O}(Y)) = \bar{A}$ holds. Then $F = H/G$ follows, and the germ $F_{p(a)}$ is the germ of a holomorphic function at $p(a)$, since the germ $G_{p(a)}$ of G at $p(a)$ is a unit. The surjectivity of p implies that F is holomorphic on Y, completing the proof of the theorem.

Note that the topology induced by $\mathcal{O}(\tilde{X})$ on any subalgebra A of $\mathcal{O}(\tilde{X})$ is the metrizable topology of uniform convergence on compact subsets of X. Because the closure $\bar{A}$ of A in $\mathcal{O}(\tilde{X})$ is its completion, $\bar{A}$ can be obtained without referring directly to $\mathcal{O}(\tilde{X})$. Thus the Main Theorem can be stated as follows:

If $\tilde{X}$ denotes the normalization of an irreducible Stein space X, then $\mathcal{O}(\tilde{X})$ is the completion of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$.

3. Applications

In this section X will denote an irreducible Stein space with normalization $\pi: \tilde{X} \to X$, $\widetilde{\mathcal{O}(X)}$ will be the integral closure of the holomorphic functions $\mathcal{O}(X)$ on X, $\tilde{\mathcal{O}}(X)$ the Fréchet algebra of weakly holomorphic functions on X (or equivalently, the Fréchet algebra of holomorphic functions $\mathcal{O}(\tilde{X})$ on $\tilde{X}$), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \qquad \text{for} \qquad x \in X \,.$$

Although the example given in the first section shows that the algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are not always equal, the inclusion $(*)$ in the proof of the Main Theorem implies that they are locally equal in the following sense.

THEOREM 2. *For every $x \in X$, the localizations of $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ with respect to S_x coincide.*

The next theorem implies an algebraic description of the topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$.

THEOREM 3. $\mathcal{O}(\tilde{X}) = \bigcap_{x \in X} S_x^{-1} \widetilde{\mathcal{O}(X)}$.

Proof. Let $f \in M(\tilde{X}) = M(X)$ be such that for every $x \in X$ there is a $g \in S_x$ and an $h \in \widetilde{\mathcal{O}(X)}$, with $f = h/g \circ \pi$. Then the germ f_a of f at an arbitrary point $a \in \tilde{X}$ is holomorphic, because the germ of $g \circ \pi$ at a is a unit. Hence $f \in \mathcal{O}(\tilde{X})$, and the assertion is proved.

COROLLARY 2. The topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$ is the intersection of the localizations of $\widetilde{\mathcal{O}(X)}$ with respect to S_x for all $x \in X$.

The next result characterizes the weakly holomorphic functions on X as being exactly those meromorphic functions on X which are almost integral over $\mathcal{O}(X)$.

COROLLARY 3. *$\mathcal{O}(\tilde{X})$ is completely normal.*

Proof. Let $f \in M(\tilde{X})$ be almost integral over $\mathcal{O}(\tilde{X})$. Then f is almost integral over $\mathcal{O}(X)$ and therefore over $S_x^{-1} \widetilde{\mathcal{O}(X)}$ for every $x \in X$ which has been shown to be completely normal in the proof of the Main Theorem. An application of Theorem 3 yields $f \in \mathcal{O}(\tilde{X})$ and hence the assertion.

Using the classical Oka-Weil-Cartan Theorem [1, Anhang zu VI, § 4], an immediate consequence of the Main Theorem is

THEOREM 4. *$\tilde{X}$ is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$.*

A property which ensures that the holomorphic functions on $\tilde{X}$ are integral over the holomorphic functions on X is that $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$-module.

THEOREM 5. *Let $u \in \mathcal{O}(X)$ be any global universal denominator for X. Then $\mathcal{O}(\tilde{X})$ is isomorphic to the closed ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$, and $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$-module if and only if this ideal is finitely generated.*

Proof. Recall that a global universal denominator u for X always exists [10, E.73a]. The multiplication map

$$\mathcal{O}(\tilde{X}) \to \mathcal{O}(X), \ f \mapsto uf,$$

defines an injective $\mathcal{O}(X)$-module homomorphism. Thus, $\mathcal{O}(\tilde{X})$ is isomorphic to the ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$ which will now be denoted by I. Consider the transporter ideal $J := \tilde{\mathcal{O}} : \frac{1}{u}\mathcal{O}$ of $\frac{1}{u}\mathcal{O}$ into $\tilde{\mathcal{O}}$ which is a coherent sheaf of ideals in $\tilde{\mathcal{O}}$. The global sections $J(X)$ form a closed ideal of $\mathcal{O}(X)$ by a theorem of Cartan [4, 5], due again to the fact that X is Stein. Because $J(X) = I$ holds, the assertion follows.

COROLLARY 4. *If* $\mathcal{O}(\tilde{X})$ *does not coincide with* $\widetilde{\mathcal{O}(X)}$, *the closed ideal* $u\mathcal{O}(\tilde{X})$ *in* $\mathcal{O}(X)$ *is not finitely generated.*

In a Stein algebra $\mathcal{O}(X)$, every finitely generated ideal is closed, as Cartan [4, 5] showed. If X is at least two-dimensional, Forster [6] gave examples of closed ideals in $\mathcal{O}(X)$ which are not finitely generated. According to Corollary 4, the space constructed in § 1 gives a one-dimensional example.

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