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GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

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INTRODUCTION

A fundamental tool in the theory of complex manifolds X is Riemann's Theorem on Removable Singularities of holomorphic functions which ensures that all functions holomorphic outside of a rare analytic subset of X and locally bounded on X can be extended to functions holomorphic on all of X. In other words, all weakly holomorphic functions on X are actually holomorphic. Although this theorem does not hold for arbitrary complex spaces, Oka [12] showed in 1951 that every complex space X can be modified to a complex space $\tilde{X}$ for which Riemann's continuation theorem is valid, the so-called normalization of X.

Stein spaces X are complex spaces which can be completely described by the algebra $\mathcal{O}(X)$ of global holomorphic functions. Since a complex space is Stein if and only if its normalization is Stein [11], it is natural to ask if the normalization $\tilde{X}$ of a Stein space X can be constructed just from the holomorphic functions on X. Phrased differently, the question is whether the algebra $\mathcal{O}(\tilde{X})$ of all holomorphic functions on $\tilde{X}$ or equivalently, the algebra $\tilde{\mathcal{O}}(X)$ of all weakly holomorphic functions on X, can be derived from the algebra $\mathcal{O}(X)$ of holomorphic functions on X.

The purpose of this paper is to demonstrate that this is possible when X is irreducible: $\tilde{\mathcal{O}}(X)$ is the topological closure of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$. An example given in § 1 shows that $\widetilde{\mathcal{O}(X)}$ is not in general topologically closed even if X is locally irreducible. $\tilde{\mathcal{O}}(X)$ can also be obtained by taking the intersection of the localizations $S_x^{-1}\widetilde{\mathcal{O}(X)}$ of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ with respect to $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$ for every $x \in X$ (see § 3).

The proof relies on an analytic and an algebraic theorem, namely Rossi's theorem [13] generalizing the Remmert quotient and the integral closure theorem of Mori-Nagata [7].

An analytic consequence of the construction presented here is that the normalization $\tilde{X}$ of an irreducible Stein space X is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$. Some algebraic results are that $\mathcal{O}(\tilde{X})$ is completely normal and that the two algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are always locally equal, i.e. their localizations at all maximal ideals in $\mathcal{O}(X)$ are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

1. Example of a Stein space X with $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let $(X, \mathcal{O})$ be a complex space with normalization $\pi: \tilde{X} \to X$. Since π is surjective, the map $\pi^*: \mathcal{O}(X) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ \pi$, is injective and the holomorphic functions $\mathcal{O}(X)$ on X can be considered to be a subring of the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of X; this will be indicated by $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$. If X is irreducible and Stein, then $\mathcal{O}(\tilde{X})$ contains the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ but does not always coincide with it, as will be shown in this section.

For an irreducible complex space $(X, \mathcal{O})$, the integral domain $\mathcal{O}(X)$ is said to be *normal*, if it is integrally closed in its field of fractions $Q(\mathcal{O}(X))$, i.e. $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$. Recall that $Q(\mathcal{O}(X))$ is the field of meromorphic functions $M(X)$ on X when X is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras $M(X)$ and $M(\tilde{X})$ are isomorphic for every complex space X [8, p. 161].

The following characterization of normal irreducible Stein spaces X by their global function algebra $\mathcal{O}(X)$ is essentially contained in [2, § 1, p. 35].

Theorem 1. *An irreducible Stein space X is normal if and only if the integral domain $\mathcal{O}(X)$ is normal.*

An analysis of the proof shows that even when X is just irreducible and normal, $\mathcal{O}(X)$ is also normal. Theorem 1 implies

Corollary 1. *For an irreducible Stein space X with normalization $\tilde{X}$, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is contained in $\mathcal{O}(\tilde{X})$.*

The following example shows that there are functions $f \in \mathcal{O}(\tilde{X})$ which are not integral over $\mathcal{O}(X)$. In this example, $X := (\mathbf{C}, \mathcal{O}')$ is an irreducible