**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 34 (1988)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

**Autor:** Hayes, Sandra / Pourcin, Geneviève

**DOI:** https://doi.org/10.5169/seals-56603

**Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

**Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

**Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

**Download PDF:** 07.01.2026

**ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch**

*L'Enseignement Mathématique, t. 34 (1988), p. 357-364*

# GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

by Sandra HAYES and Geneviève POURCIN

## INTRODUCTION

A fundamental tool in the theory of complex manifolds $X$ is Riemann's Theorem on Removable Singularities of holomorphic functions which ensures that all functions holomorphic outside of a rare analytic subset of $X$ and locally bounded on $X$ can be extended to functions holomorphic on all of $X$. In other words, all weakly holomorphic functions on $X$ are actually holomorphic. Although this theorem does not hold for arbitrary complex spaces, Oka [12] showed in 1951 that every complex space $X$ can be modified to a complex space $\tilde{X}$ for which Riemann's continuation theorem is valid, the so-called normalization of $X$.

Stein spaces $X$ are complex spaces which can be completely described by the algebra $\mathcal{O}(X)$ of global holomorphic functions. Since a complex space is Stein if and only if its normalization is Stein [11], it is natural to ask if the normalization $\tilde{X}$ of a Stein space $X$ can be constructed just from the holomorphic functions on $X$. Phrased differently, the question is whether the algebra $\mathcal{O}(\tilde{X})$ of all holomorphic functions on $\tilde{X}$ or equivalently, the algebra $\tilde{\mathcal{O}}(X)$ of all weakly holomorphic functions on $X$, can be derived from the algebra $\mathcal{O}(X)$ of holomorphic functions on $X$.

The purpose of this paper is to demonstrate that this is possible when $X$ is irreducible: $\tilde{\mathcal{O}}(X)$ is the topological closure of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$. An example given in § 1 shows that $\widetilde{\mathcal{O}(X)}$ is not in general topologically closed even if $X$ is locally irreducible. $\tilde{\mathcal{O}}(X)$ can also be obtained by taking the intersection of the localizations $S_x^{-1}\,\widetilde{\mathcal{O}(X)}$ of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ with respect to $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$ for every $x \in X$ (see § 3).

The proof relies on an analytic and an algebraic theorem, namely Rossi's theorem [13] generalizing the Remmert quotient and the integral closure theorem of Mori-Nagata [7].

An analytic consequence of the construction presented here is that the normalization $\tilde{X}$ of an irreducible Stein space $X$ is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$. Some algebraic results are that $\mathcal{O}(\tilde{X})$ is completely normal and that the two algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are always locally equal, i.e. their localizations at all maximal ideals in $\mathcal{O}(X)$ are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

## 1. Example of a Stein space $X$ with $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let $(X, \mathcal{O})$ be a complex space with normalization $\pi: \tilde{X} \to X$. Since $\pi$ is surjective, the map $\pi^*: \mathcal{O}(X) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ \pi$, is injective and the holomorphic functions $\mathcal{O}(X)$ on $X$ can be considered to be a subring of the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of $X$; this will be indicated by $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$. If $X$ is irreducible and Stein, then $\mathcal{O}(\tilde{X})$ contains the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ but does not always coincide with it, as will be shown in this section.

For an irreducible complex space $(X, \mathcal{O})$, the integral domain $\mathcal{O}(X)$ is said to be *normal*, if it is integrally closed in its field of fractions $Q(\mathcal{O}(X))$, i.e. $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$. Recall that $Q(\mathcal{O}(X))$ is the field of meromorphic functions $M(X)$ on $X$ when $X$ is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras $M(X)$ and $M(\tilde{X})$ are isomorphic for every complex space $X$ [8, p. 161].

The following characterization of normal irreducible Stein spaces $X$ by their global function algebra $\mathcal{O}(X)$ is essentially contained in [2, § 1, p. 35].

Theorem 1. *An irreducible Stein space $X$ is normal if and only if the integral domain $\mathcal{O}(X)$ is normal.*

An analysis of the proof shows that even when $X$ is just irreducible and normal, $\mathcal{O}(X)$ is also normal. Theorem 1 implies

Corollary 1. *For an irreducible Stein space $X$ with normalization $\tilde{X}$, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is contained in $\mathcal{O}(\tilde{X})$.*

The following example shows that there are functions $f \in \mathcal{O}(\tilde{X})$ which are not integral over $\mathcal{O}(X)$. In this example, $X := (\mathbf{C}, \mathcal{O}')$ is an irreducible

and locally irreducible Stein space given by a substructure of the canonical complex plane $(\mathbf{C}, \mathcal{O})$, which is then the normalization $\tilde{X}$ of $X$. The substructure is defined by a "Strukturausdünnung" (see [10]) which results by replacing the stalks $\mathcal{O}_n$, $n \in \mathbf{N}$, with the stalks of generalized Neil parabolas becoming steeper as $n$ increases. More precisely, let $(p_n)_{n\in\mathbf{N}}$ be a strictly increasing sequence of prime numbers. For every $n \in \mathbf{N}$,

$$X_n := \{(x, y) \in \mathbf{C}^2 : x^{p_n} = y^{p_n+1}\}$$

is an irreducible, locally irreducible analytic subset of $\mathbf{C}^2$ with the origin as the only singularity and with normalization

$$\pi_n : \mathbf{C} \to X_n, \quad t \mapsto (t^{p_n+1}, t^{p_n}).$$

Let $f \in \mathcal{O}(\mathbf{C})$ be the identity and denote by $\mathcal{O}_{X_n}$ the canonical complex structure on $X_n$. The germ $f_0 \in \mathcal{O}_0$ of $f$ at the origin is integral over $\mathcal{O}_{X_{n,0}}$ with respect to a polynomial of degree $p_n$, and $p_n$ is the minimal degree of all such polynomials.

Now define $X := (\mathbf{C}, \mathcal{O}')$ as a substructure of the canonical plane $(\mathbf{C}, \mathcal{O})$ with stalks

$$\mathcal{O}'_x \cong \begin{cases} \mathcal{O}_x, & x \notin \mathbf{N} \\ \mathcal{O}_{X_{n,0}}, & x = n \in \mathbf{N} \end{cases}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}'_n & \to & \mathcal{O}_n \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{O}_{X_{n,0}} & \xrightarrow{\pi_n^*} & \mathcal{O}_0, \end{array}$$

where $\mathcal{O}'_n \to \mathcal{O}_n$ is the map induced by the identity $(\mathbf{C}, \mathcal{O}) \to (\mathbf{C}, \mathcal{O}')$ and $\mathcal{O}_n \cong \mathcal{O}_0$ is determined by the translation $\mathbf{C} \to \mathbf{C}$, $z \mapsto z - n$.

The identity $f \in \mathcal{O}(\mathbf{C})$ is not integral over $\mathcal{O}'(\mathbf{C})$, because otherwise every polynomial of integral dependence would have degree at least $p_n$ for all $n \in \mathbf{N}$.

In conclusion it should be mentioned that $\mathcal{O}(\tilde{X})$ is almost integral over $\mathcal{O}(X)$ [7, § 3] for every irreducible Stein space $X$, since $X$ has a global universal denominator [10, E.73a].

## 2. Construction of $\mathcal{O}(\tilde{X})$ from $\mathcal{O}(X)$ for Stein spaces $X$

According to a theorem of Oka [12], the normalization sheaf $\tilde{\mathcal{O}}$ of weakly holomorphic functions on a complex space $(X, \mathcal{O})$ is coherent. Consequently, there is a canonical topology making $\tilde{\mathcal{O}}$ a Fréchet sheaf; the global weakly holomorphic functions $\tilde{\mathcal{O}}(X)$ will always carry this topology. Since the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of $X$ are topologically isomorphic to $\tilde{\mathcal{O}}(X)$ [8, 8.3], the question posed in the introduction can now be answered.

Main theorem. *For an irreducible Stein space $X$, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is dense in $\tilde{\mathcal{O}}(X)$.*

*Proof.* Let $\pi: \tilde{X} \to X$ be the normalization of $X$ and put $A := \widetilde{\mathcal{O}(X)}$. Since $\pi$ is proper, $\tilde{X}$ is $\mathcal{O}(X)$-convex and therefore $\bar{A}$-convex. Note that Corollary 1 implies $A \subset \tilde{\mathcal{O}}(X)$ and that $\bar{A}$ is the closure of $A$ with respect to the canonical topology in $\tilde{\mathcal{O}}(X)$.

Consider the equivalence relation $R$ on $\tilde{X}$ defined by $\bar{A}$, i.e. $(x, y) \in R$ iff for every $f \in \bar{A}$, $f(x) = f(y)$. Rossi's theorem [13] ensures that the topological quotient $Y := \tilde{X}/R$ can be given the complex structure of a Stein space such that the projection $p: \tilde{X} \to Y$ is holomorphic and proper and the map $p^*: \mathcal{O}(Y) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ p$, induces an isomorphism $\mathcal{O}(Y) \cong \bar{A}$.

It suffices to show that every $f \in \mathcal{O}(\tilde{X})$ can be factorized through a holomorphic function on $Y$, meaning that an $F \in \mathcal{O}(Y)$ exists with $F \circ p = f$. This will be accomplished by first factorizing $f \in \mathcal{O}(\tilde{X})$ through a continuous function $F$ on $Y$ and then proving that $F$ is actually holomorphic. The existence of such a continuous factor $F$ for $f$ is equivalent to demonstrating that every $f \in \mathcal{O}(\tilde{X})$ is constant on the fibers of $p$. The validity of this geometric statement will be shown now using commutative algebra.

$\mathcal{O}(\tilde{X})$ is almost integral over $\mathcal{O}(X)$ (see § 1), and hence over the localization $S_x^{-1}A$ of $A$ with respect to $S_x := \{g \in \mathcal{O}(X): g(x) \neq 0\}$ for every $x \in X$. Moreover,

$$S_x^{-1}A = \widetilde{S_x^{-1}\mathcal{O}(X)}$$

holds [3, V, 1.5, Corollary 1]. The localization $S_x^{-1}\mathcal{O}(X) = \mathcal{O}(X)_{m(x)}$ of the Stein algebra $\mathcal{O}(X)$ at the maximal ideal $m(x) := \{f \in \mathcal{O}(X): f(x) = 0\}$ is noetherian — even more, it's excellent [2, p. 35]. According to a theorem of Mori-Nagata, the integral closure of a noetherian integral domain is completely normal [7, 4.3, 3.6], implying

$$(*) \qquad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A .$$

For $f \in \mathcal{O}(\tilde{X})$, $a \in \tilde{X}$ and $b \in p^{-1}(p(a))$, it is now possible to conclude that $f(a) = f(b)$ is true. Let $x := \pi(a)$. Due to $(*)$, functions $g \in S_x$ and $h \in A$ exist with $f = h/g \circ \pi$. Since $a$ and $b$ are equivalent with respect to the equivalence relation $R$, $f(a) = f(b)$ follows, and a continuous function $F: Y \to \mathbf{C}$ exists with $F \circ p = f$.

Since the Stein complex structure on $Y$ is not in general the canonical ringed quotient structure, it is still necessary to verify that $F$ is holomorphic in order to prove the density of $A$ in $\mathcal{O}(\tilde{X})$. To that end, let $H \in \mathcal{O}(Y)$ and $G \in \mathcal{O}(Y)$ have the property that $H \circ p = h$ and $G \circ p = g \circ \pi$. Such functions exist because $p^*(\mathcal{O}(Y)) = \bar{A}$ holds. Then $F = H/G$ follows, and the germ $F_{p(a)}$ is the germ of a holomorphic function at $p(a)$, since the germ $G_{p(a)}$ of $G$ at $p(a)$ is a unit. The surjectivity of $p$ implies that $F$ is holomorphic on $Y$, completing the proof of the theorem.

Note that the topology induced by $\mathcal{O}(\tilde{X})$ on any subalgebra $A$ of $\mathcal{O}(\tilde{X})$ is the metrizable topology of uniform convergence on compact subsets of $X$. Because the closure $\bar{A}$ of $A$ in $\mathcal{O}(\tilde{X})$ is its completion, $\bar{A}$ can be obtained without referring directly to $\mathcal{O}(\tilde{X})$. Thus the Main Theorem can be stated as follows:

*If $\tilde{X}$ denotes the normalization of an irreducible Stein space $X$, then $\mathcal{O}(\tilde{X})$ is the completion of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$.*

## 3. Applications

In this section $X$ will denote an irreducible Stein space with normalization $\pi: \tilde{X} \to X$, $\widetilde{\mathcal{O}(X)}$ will be the integral closure of the holomorphic functions $\mathcal{O}(X)$ on $X$, $\tilde{\mathcal{O}}(X)$ the Fréchet algebra of weakly holomorphic functions on $X$ (or equivalently, the Fréchet algebra of holomorphic functions $\mathcal{O}(\tilde{X})$ on $\tilde{X}$), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \qquad \text{for} \qquad x \in X .$$

Although the example given in the first section shows that the algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are not always equal, the inclusion $(*)$ in the proof of the Main Theorem implies that they are locally equal in the following sense.

THEOREM 2. *For every $x \in X$, the localizations of $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ with respect to $S_x$ coincide.*

The next theorem implies an algebraic description of the topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$.

THEOREM 3. $\mathcal{O}(\tilde{X}) = \bigcap_{x \in X} S_x^{-1} \widetilde{\mathcal{O}(X)}$.

*Proof.* Let $f \in M(\tilde{X}) = M(X)$ be such that for every $x \in X$ there is a $g \in S_x$ and an $h \in \widetilde{\mathcal{O}(X)}$, with $f = h/g \circ \pi$. Then the germ $f_a$ of $f$ at an arbitrary point $a \in \tilde{X}$ is holomorphic, because the germ of $g \circ \pi$ at a is a unit. Hence $f \in \mathcal{O}(\tilde{X})$, and the assertion is proved.

COROLLARY 2. The topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$ is the intersection of the localizations of $\widetilde{\mathcal{O}(X)}$ with respect to $S_x$ for all $x \in X$.

The next result characterizes the weakly holomorphic functions on $X$ as being exactly those meromorphic functions on $X$ which are almost integral over $\mathcal{O}(X)$.

COROLLARY 3. *$\mathcal{O}(\tilde{X})$ is completely normal.*

*Proof.* Let $f \in M(\tilde{X})$ be almost integral over $\mathcal{O}(\tilde{X})$. Then $f$ is almost integral over $\mathcal{O}(X)$ and therefore over $S_x^{-1} \widetilde{\mathcal{O}(X)}$ for every $x \in X$ which has been shown to be completely normal in the proof of the Main Theorem. An application of Theorem 3 yields $f \in \mathcal{O}(\tilde{X})$ and hence the assertion.

Using the classical Oka-Weil-Cartan Theorem [1, Anhang zu VI, § 4], an immediate consequence of the Main Theorem is

THEOREM 4. *$\tilde{X}$ is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$.*

A property which ensures that the holomorphic functions on $\tilde{X}$ are integral over the holomorphic functions on $X$ is that $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$-module.

THEOREM 5. *Let $u \in \mathcal{O}(X)$ be any global universal denominator for $X$. Then $\mathcal{O}(\tilde{X})$ is isomorphic to the closed ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$, and $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$-module if and only if this ideal is finitely generated.*

*Proof.* Recall that a global universal denominator $u$ for $X$ always exists [10, E.73a]. The multiplication map

$$\mathcal{O}(\tilde{X}) \to \mathcal{O}(X), \; f \mapsto uf,$$

defines an injective $\mathcal{O}(X)$-module homomorphism. Thus, $\mathcal{O}(\tilde{X})$ is isomorphic to the ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$ which will now be denoted by $I$. Consider the transporter ideal $J := \tilde{\mathcal{O}} : \frac{1}{u}\mathcal{O}$ of $\frac{1}{u}\mathcal{O}$ into $\tilde{\mathcal{O}}$ which is a coherent sheaf of ideals in $\tilde{\mathcal{O}}$. The global sections $J(X)$ form a closed ideal of $\mathcal{O}(X)$ by a theorem of Cartan [4, 5], due again to the fact that $X$ is Stein. Because $J(X) = I$ holds, the assertion follows.

COROLLARY 4. *If $\mathcal{O}(\tilde{X})$ does not coincide with $\widetilde{\mathcal{O}(X)}$, the closed ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$ is not finitely generated.*

In a Stein algebra $\mathcal{O}(X)$, every finitely generated ideal is closed, as Cartan [4, 5] showed. If $X$ is at least two-dimensional, Forster [6] gave examples of closed ideals in $\mathcal{O}(X)$ which are not finitely generated. According to Corollary 4, the space constructed in § 1 gives a one-dimensional example.

## REFERENCES


[1] BEHNKE, H. und P. THULLEN. *Theorie der Funktionen mehrerer komplexen Veränderlichen*, 2. edition. Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[2] BINGENER, J. und U. STORCH. Resträume zu analytischen Mengen in Steinschen Räumen. *Math. Ann. 210* (1974), 33-53.

[3] BOURBAKI, N. *Algèbre commutative*. Hermann, Paris, 1969.

[4] CARTAN, H. *Séminaire*. E.N.S. 1951/1952.

[5] —— Idéaux et modules de fonctions analytiques de variables complexes. *Bull. Soc. Math. France 78* (1950), 28-64.

[6] FORSTER, O. Zur Theorie der Steinschen algebren und Moduln. *Math. Zeitschr. 97* (1967), 376-405.

[7] FOSSUM, R. *The divisor class group of a Krull domain*. Ergebnisse der Math. und ihrer Grenzgebiete 74, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[8] GRAUERT, H. and R. REMMERT. *Coherent analytic sheaves*. Grundl. Math. Wiss. 265. Springer-Verlag, Berlin, Heidelberg, New York, 1984.

[9] —— *Theorie der Steinschen Räume*. Grundl. Math. Wiss. 227. Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[10] KAUP, L. and B. KAUP. *Holomorphic functions of several variables.* De Gruyter, New York, 1983.

[11] NARASIMHAN, R. A note on Stein spaces and their normalizations. *Ann. Scuola Norm. Sup. Pisa 16* (1962), 327-333.

[12] OKA, K. Sur les fonctions analytiques de plusieurs variables, VIII, Lemme fondamental. *J. Math. Soc. Japan 3* (1951), 204-214, 259-278.

[13] ROSSI, H. Analytic spaces with compact subvarieties. *Math. Ann. 146* (1962), 129-145.

*(Reçu le 13 mars 1988)*

Sandra Hayes
Mathematisches Institut
Technische Universität
D-8000 München 2
(Federal Republic of Germany)

Geneviève Pourcin
Faculté des Sciences
2, Boulevard Lavoisier
F-49045 Angers
(France)