

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	34 (1988)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	SOME ALMOST HOMOGENEOUS GROUP ACTIONS ON SMOOTH COMPLETE RATIONAL SURFACES
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<b>Kapitel:</b>	§3. Application to $SL(2)$ -embeddings
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-56601">https://doi.org/10.5169/seals-56601</a>

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This finishes Case 3. Thus we know all the embeddings into  $\mathbf{P}^2$ ,  $\mathbf{P}^1 \times \mathbf{P}^1$  and  $\mathbf{F}_n$ ,  $n \geq 1$ . The comments after Theorem 2.1 are easily verified by checking each embedding. This finishes the proof of the theorem.  $\square$

*Remarks.*

(1) Note that — as to be expected — all the embeddings into  $\mathbf{F}_1$  are obtained by blowing up the embeddings into  $\mathbf{P}^2$  at fixed points.

(2) The “exceptional” embeddings, i.e. those with only one fixed point, are of special interest, because this phenomenon does not occur for smooth complete embeddings of tori. (See [KKMS] for a reference on torus embeddings.)

### § 3. APPLICATION TO $SL(2)$ -EMBEDDINGS

In [LV] a combinatorical method is presented in order to classify all normal  $SL(2)$ -embeddings. A natural question is how to classify those which are smooth and complete to obtain a *geometrical* realization. We now sketch how the result of this article is useful for this. (For further details see [JM].)

Given a  $B/\Gamma$ -embedding  $X$ , we construct an  $SL(2)/\Gamma$ -embedding in the following way. Consider the  $B$ -action on  $SL(2) \times X$  given by

$$b \cdot (s, x) = (sb^{-1}, bx)$$

where  $b \in B$ ,  $s \in SL(2)$ , and  $x \in X$ . Denote by  $SL(2)*_B X$  the variety obtained by quotienting by this action. The action of  $SL(2)$  on this variety by left multiplication endows it with the structure of an  $SL(2)/\Gamma$ -embedding. The projection  $SL(2) \times X \rightarrow SL(2)$  induces a locally trivial fibre bundle  $SL(2)*_B X \xrightarrow{p} SL(2)/B \cong \mathbf{P}^1$ . The morphism  $p$  is  $SL(2)$ -equivariant, and the fibre of  $p$  is  $B$ -isomorphic to  $X$ . So we see that for studying the geometry of the  $SL(2)/\Gamma$ -embeddings of this form it is useful to study the  $B/\Gamma$ -embeddings.

As for general  $SL(2)/\Gamma$ -embeddings one finds the following essential result. Let  $\Gamma$  be a finite cyclic subgroup of  $SL(2)$ . Let  $V$  be a smooth  $SL(2)/\Gamma$ -embedding with orbit  $Y$ . Then there exists a Borel subgroup  $B$  of  $SL(2)$  containing  $\Gamma$  and an  $SL(2)$ -stable open neighborhood of  $Y$  in  $V$  which is of the form  $SL(2)*_B X$  for some smooth  $B/\Gamma$ -embedding  $X$ . Thus all smooth  $SL(2)/\Gamma$ -embeddings are *locally* of the form above. Also any smooth  $B/\Gamma$ -embedding can be completed to a smooth embedding. Thus it is enough to study the complete ones.

We can use this fact, for example, to study blow-ups of orbits, since blowing up is a local property. Thus we can find the minimal  $SL(2)/\Gamma$ -embeddings. This is done in [JM], Chapter IV, for  $\Gamma = \{e\}$  and  $\Gamma = \{\pm e\}$ .

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(Reçu le 23 février 1988)

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