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on the spin bundle S_+ is anti-self-dual. Recall (see § 3) that for Γ Fuchsian, extended Fuchsian or a suitable Schottky group X_Γ admits such a metric. The connection on S_+ is a monopole because the metrics are S^1-invariant. The mass(es) is (are) 1 by proposition 2.2, and the charges k_j equal $g - 1$, where g is the genus of the fixed surface(s). Choosing a different spin structure amounts to tensoring the bundle with a 2-torsion element in $\mathrm{Repr}\,(\pi_1(M), S^1)$, compare 2).

In section 7 we shall come to grips with explicit formulae for nontrivial monopoles on certain handlebodies. In Braam-Hurtubise [11] the moduli spaces of monopoles on a solid torus are investigated in considerable detail. A general existence theory for monopoles on hyperbolic manifolds has been developed in Braam [10].

§ 6. Twistor spaces

To a conformally flat oriented 4-manifold X there are naturally associated two complex manifolds Z_+ and Z_-, the *twistor spaces* of X. Applying our construction of § 2 we thus get twistor spaces for hyperbolic 3-manifolds. It will be shown here that these carry a lot of geometric information associated to the 3-manifold M, such as the complete geodesic flow. Also they allow for a description of monopoles through holomorphic geometry. For the rest of this section let X be the conformal compactification of $M \times S^1$, with M a hyperbolic 3-manifold H^3/Γ as in § 2. We shall state those properties of $Z_\pm$ that we will need, and refer to Atiyah [1] and Atiyah-Hitchin-Singer [5] for proofs and more details. The general line of thought in this section is very similar to that of Hitchin [20] and Atiyah [2].

If $S_+(S_-)$ is the spin bundle of positive (negative) chirality on X, then $Z_+(Z_-)$ can be realised as the $\mathbf{CP}^1$-bundles over X:

$$P(S_+) \to X \qquad (P(S_-) \to X)\,,$$

where $P(\ )$ denotes projectivization of vectorbundles. A remarkable fact is that Z_+ and Z_- are *complex manifolds* with a complex structure encoded in the conformal structure of X. However, the twistor spaces are only Kähler if $X \cong S^4$ or $X \cong \mathbf{CP}^2$, which in our case results in $\Gamma = \{e\}$ (see Hitchin [19]). There is an orientation reversing isometry of X arising from conjugation of the circles. This interchanges the two spin bundles and makes

Z_+ holomorphically equivalent to Z_-. Henceforth we shall only consider Z_+ and denote it by Z.

Z carries an *anti-holomorphic involution*:

$$\sigma: Z \to Z, \quad \sigma^2 = 1.$$

This involution is a bundle map, inducing the identity on the base X, and is equal to the antipodal map upon restriction to the fibres. The complex structure on Z is such that (orientation preserving) conformal transformations on X lift to holomorphic transformations of Z. So our S^1-action on X lifts to an action on Z by holomorphic transformations and complexifies to a holomorphic $\mathbf{C}^*$-action on Z. We shall show that this $\mathbf{C}^*$-action is essentially the *geodesic flow* in H^3/Γ (as one would expect from Hitchin [20]).

The naturality with respect to conformal transformations has one further important application.

Recall (see Atiyah [1]) that the twistor space of S^4 is $\mathbf{CP}^3$ with projection and real structure:

$$\pi: \mathbf{CP}^3 \to S^4 = \mathbf{HP}^1: [z_0, z_1, z_2, z_3] \to [z_0 + z_1 \cdot j, z_2 + z_3 \cdot j]$$
$$\sigma: \mathbf{CP}^3 \to \mathbf{CP}^3: [z_0, z_1, z_2, z_3] \to [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2]$$

As $X = (S^4 - \Lambda)/\Gamma$ it follows that the twistor space of X is the quotient:

$$Z = [\mathbf{CP}^3 - \pi^{-1}(\Lambda)]/\Gamma.$$

To study Z it will be useful to know how $\mathbf{C}^*$ and $PSL(2, \mathbf{C})$ act on $\mathbf{CP}^3$. The $\mathbf{C}^*$ action is described by $[z_0, z_1, z_2, z_3] \to [z_0, \lambda \cdot z_1, z_2, \lambda \cdot z_3]$, and

the right $PSL(2, \mathbf{C})$-action by mapping $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to $\begin{bmatrix} a & 0 & c & 0 \\ 0 & \bar{a} & 0 & \bar{c} \\ b & 0 & d & 0 \\ 0 & \bar{b} & 0 & \bar{d} \end{bmatrix} \in PSL(4, \mathbf{C})$

which acts naturally on $\mathbf{CP}^3$, compare 2.3. Clearly the S^1-action fixes precisely two lines in $\mathbf{CP}^3$ namely:

6.1
$$P_1^+ = \{[z_0, 0, z_2, 0] \in \mathbf{CP}^3\} \quad \text{and}$$
$$P_1^- = \{[0, z_1, 0, z_3] \in \mathbf{CP}^3\}$$

These lines are also invariant under the hyperbolic isometries. The projections to the fixed point set $S^2 \subset S^4$ are the orientation preserving map $P_1^+ \to S^2: [z_0, z_2] \to [z_0, z_2]$ and the orientation reversing map

$P_1^- \to S^2 : [z_1, z_3] \to [\bar{z}_1, \bar{z}_3]$ respectively. Here we have used homogeneous quaternionic coordinates on $S^4 = \mathbf{HP}^1$. The real structure maps P_1^+ to P_1^- and vice versa.

Non-trivial $\mathbf{C}^*$-orbits in $\mathbf{CP}^3$ are in one-one correspondence with a pair of begin- and end-points $(z, w) \in P_1^+ \times P_1^-$. Upon projecting the orbit $\mathcal{O}$ corresponding to (z, w) down to H^3:

$$\mathcal{O} \subset \mathbf{CP}^3 \to \pi(\mathcal{O}) \subset S^4 = \overline{H^3 \times S^1} \to g(\mathcal{O}) \subset H^3$$

one easily sees that $g(\mathcal{O})$ is an oriented geodesic in H^3 from $z \in S^2 = \delta H^3$ to $\bar{w} \in S^2$. The constant geodesics at infinity are included. Further for $p \in \mathcal{O} \subset \mathbf{CP}^3$ and $\lambda \in \mathbf{C}^*$ we have that the distance of $\pi(p)$ and $\pi(\lambda p)$ on $g(\mathcal{O})$ equals $\log|\lambda|$. As the $\mathbf{C}^*$-action commutes with the Γ-action, this shows that the $\mathbf{C}^*$-action is essentially geodesic flow in M. More precisely consider a copy of $M = i(M \times \{1\})$ in X. Then $Z_{|M}$ is the projectivized spin bundle of M which is canonically isomorphic to the unit tangent sphere bundle of M. Further the action of $\mathbf{R}_{>0} \subset \mathbf{C}^*$ preserves $Z_{|M}$ and is exactly the geodesic flow.

It is now possible to describe Z in detail. First of all the fixed points of the $\mathbf{C}^*$-action on Z are surfaces S_j^+, S_j^-, which project down to $S_j \subset X$. The surfaces S_j^+, S_j^- equal the components of $[P_1^+ - \Lambda]/\Gamma$ and $[P_1^- - \Lambda]/\Gamma$ respectively. The real structure maps S_j^+ to S_j^-.

The nontrivial $\mathbf{C}^*$-orbits in Z come in three types. *Good orbits* emanate from a plus surface, say S_j^+, and end on a minus surface, say S_k^-. The closure of one of these orbits in Z is a $\mathbf{CP}^1$. Note that these orbits are not determined by their two "endpoints". This corresponds precisely to the fact that two geodesics in M may have the same two endpoints, but in between one of them may run through different loops than the other. Denote by Ω_j^+ (Ω_j^-) the pre-image in P_1^+ (P_1^-) of S_j^+ (S_j^-) under the quotient map. From the above we get the following

PROPOSITION 6.1. *The good orbits from S_j^+ to S_k^- are in one-one correspondence with oriented geodesics in $M \cong H^3/\Gamma$, which go from S_j to S_k. These have the complex analytic parameter space $[\Omega_j^+ \times \Omega_k^-]/\Gamma$, which is a holomorphic Ω_k^- bundle over S_j^+ or equivalently an Ω_j^+ bundle over S_k^-.*

Considering all good orbits emanating from S_j^+ and ending on some S_k^-, one gets that these are holomorphically parametrized by a $\bigcup_k \Omega_k^- = P_1^- - \Lambda$ bundle over S_j^+. Indeed, all orbits emanating from S_j^+ have a nice algebraic parameter space, which is equal to the projectivized holomorphic normal bundle $P(N_j^+)$ of S_j in Z. This is a $\mathbf{CP}^1$-bundle over S_j^+. The *bad orbits* correspond to geodesics in M which, in the universal cover, start in Ω_j and end in Λ. Of course similar statements hold concerning arriving geodesics and the projectivized normal bundle of S_j^-. Concerning the normal bundles we have the following

PROPOSITION 6.2. *There are injective, open, locally biholomorphic maps* $\psi_j^\pm : N_j^\pm \to Z$, *where* $N_j^\pm$ *is the holomorphic normal bundle of* $S_j^\pm$ *in* Z. *The* $\mathbf{C}^*$*-multiplication on the bundle* N_j^+ *is intertwined with the* $\mathbf{C}^*$*-action on* Z *by* ψ_j^+, *whereas* ψ_j^- *intertwines multiplication by the inverse with the* $\mathbf{C}^*$*-action on* Z. *The projectivized normal bundles* $P(N_j^+)$ $(P(N_j^-))$ *are an algebraic parameter space for all geodesics in* M *going out from (arriving at)* S_j.

Proof. This is easy for the normal bundles of P_1^+ and P_1^- in $\mathbf{CP}^3$. Because the Γ action is linear and commutes with the $\mathbf{C}^*$-action the result also holds in Z. □

Remark 6.3. 1) The relation of the normal bundles with Eichler's modules. If $\mathscr{K} \to \mathbf{CP}^1$ is the positive Hopf bundle, then $H^0(\mathbf{CP}^1, \mathscr{K}^n) = \Pi_n$ is an $SL(2, \mathbf{C})$-module, called an *Eichler module*, see Bers [7]. Hence after choice of a spin structure $\Gamma \to SL(2, \mathbf{C})$ a Γ-module (compare the discussion after proposition 2.2). A short computation shows that the normal bundle of S_j^+ in Z is isomorphic to:

$$N_j^+ = (\Omega_j^+ \times_\Gamma \bar{\Pi}_1) \otimes V_{+,j},$$

where $V_{+,j}$ is the positive spin bundle of S_j^+.

2) In general for complex submanifolds $V \subset W$ there are obstructions for locally embedding the normal bundle in a holomorphic way, see Kodaira [23].

3) It may be possible to derive the geometry of the ends of the hyperbolic manifold M from the holomorphic structure of a normal bundle of a fixed surface. It would be interesting to have a formula for the metric on an end, giving the end as a foliation by surfaces such that the foliation is invariant under geodesic flow.

Finally there are *very bad orbits*, corresponding to geodesics going from Λ to Λ in the universal cover. In M they keep spiralling around, and never find and endpoint in either direction. For example closed geodesics are among these, in fact points in non-trivial orbits have a non-trivial stabilizer iff the orbit corresponds to a closed geodesic. The $\mathbf{C}^*$-orbits in Z corresponding to closed geodesics are compact holomorphically embedded *elliptic curves* in Z. The set of very bad orbits is closed in Z, is disjoint from the S_j, and lies in the closure of the set of very good orbits. In figure 2 we have sketched the orbit situation.

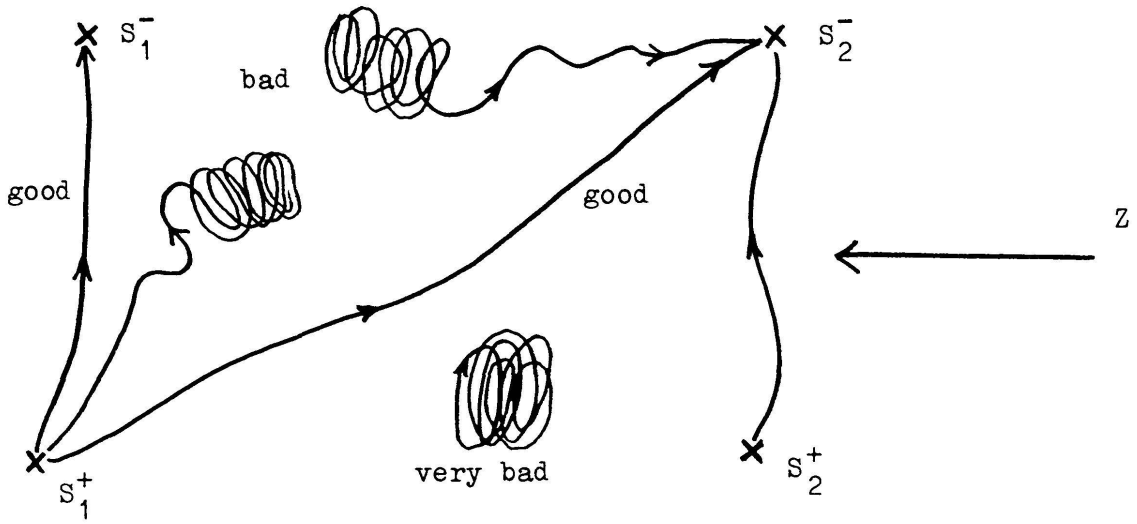

FIGURE 2.

The next objective of this section is to give a holomorphic description of monopoles. The relation between twistor spaces and anti-self-dual connections lies in the Atiyah-Ward correspondence (see Atiyah-Hitchin-Singer [5], for the instanton case):

THEOREM 6.4. *Let* $P \to X$ *be an* $\tilde{S}^1$*-equivariant* $SU(2)$*-bundle, and* A *a monopole on* P. *Put* $E = P \times_{SU(2)} \mathbf{C}^2$. *Then* $\pi^* A$ *induces a* $\tilde{\mathbf{C}}^*$*-invariant holomorphic structure on* $F = \pi^* E$ *such that:*

1) F *is trivial on the fibres of* π.

2) The natural antiholomorphic antilinear bundle map $\sigma: F \to \bar{F}^*$, *covering* σ *on* Z, *induces an* S^1*-invariant Hermitian metric on the vector spaces* $H^0(\pi^{-1}(x), F)$.

3) $\Lambda^2 F$ *is holomorphically trivial.*

Conversely a $\tilde{\mathbf{C}}^*$*-invariant holomorphic* $\mathbf{C}^2$*-bundle* F *over* Z, *with a real structure* $\sigma: F \to \bar{F}^*$ *satisfying 1, 2 and 3 arises from a unique monopole on* $P \to X$. □

Real structures on indecomposable holomorphic bundles F over twistor space are unique. Hence all the information is encoded in the holomorphic structure. However, existence of real structures is not automatic. The gauge equivalence relation for monopoles on $P \to X$ is the same as holomorphic $\tilde{\mathbf{C}}^*$-equivariant equivalence, preserving real structures, for the holomorphic bundles F on Z.

Let A be a monopole on $P \to X$, with all $m_j \neq 0$ and even, for simplicity. In this case we need not consider double coverings of groups and we shall denote the weights of S^1 by $p_j = \frac{1}{2} \cdot m_j$. Denote by $F = \pi^*(P \times_{SU(2)} \mathbf{C}^2)$ the holomorphic bundle over Z, with real structure σ. By theorem 6.4 the holomorphic structure on F is $\mathbf{C}^*$-invariant. An important aspect of monopole geometry of $\mathbf{R}^3$ and H^3 is to consider the quotient bundle $\mathscr{F} = F/\mathbf{C}^*$ on $Z/\mathbf{C}^*$ as far as this makes sense. On $Z/\mathbf{C}^*$, $\mathscr{F}$ will be an extension of certain standard line bundles, and this has been put to constructive use in the $\mathbf{R}^3$ case, see Hitchin [20]. It will be shown that a more complicated but essentially similar picture persists in our more general case. As yet, the constructive power seems to be rather limited.

Restricting F to S_j^+ it splits holomorphically, since the $\mathbf{C}^*$ action is fibre-wise, with nonzero weights $\pm p_j$:

6.2
$$F_{|S_j^+} = L_j^+ \oplus (L_j^+)^*$$
$$F_{|S_j^-} = L_j^- \oplus (L_j^-)^*$$

Here L_j^+ has $\mathbf{C}^*$-weight p_j and $c_1(L_j^+) = -k_j$, as in § 5. For L_j^- we have $\mathbf{C}^*$-weight $-p_j$ and $c_1(L_j^-) = -k_j$. The real structure gives an anti-linear isomorphism $L_j^+ \to L_j^-$.

Proposition 6.5. *On $N_j^+ \subset Z(N_j^- \subset Z)$ there are line bundles $K_j^+(K_j^-)$, extending the $L_j^\pm$ of 6.2 (which were defined on the zero sections $S_j^\pm$ of $N_j^\pm$), such that on the $N_j^\pm$ the bundle F is an extension:*

$$0 \to K_j^+ \to F_{|N_j^+} \to (K_j^+)^* \to 0$$
$$0 \to K_j^- \to F_{|N_j^-} \to (K_j^-)^* \to 0$$

The real structure interchanges these two extensions.

Proof. Recall that sections of $P(F)$ correspond to line sub-bundles of F. We shall look at the $\mathbf{C}^*$-action on $P(F)$ restricted to the fibres $(N_j^+)_z$ with $z \in S_j^+$. Over $(N_j^+)_z$ we have two fixed points in $P(F)$ namely $[(L_j^+)_z]$ and $[(L_j^+)_z^*]$, lying in the fibre above $0 \in (N_j^+)_z$. At $f = [(L_j^+)_z]$ the weights of the infinitesimal $\mathbf{C}^*$-action on $T_f P(F)$ are $(+1, +1, -p_j)$. This means that most of the $\mathbf{C}^*$-orbits will actually flow to $[(L_j^+)_z^*]$, compare figure 3.

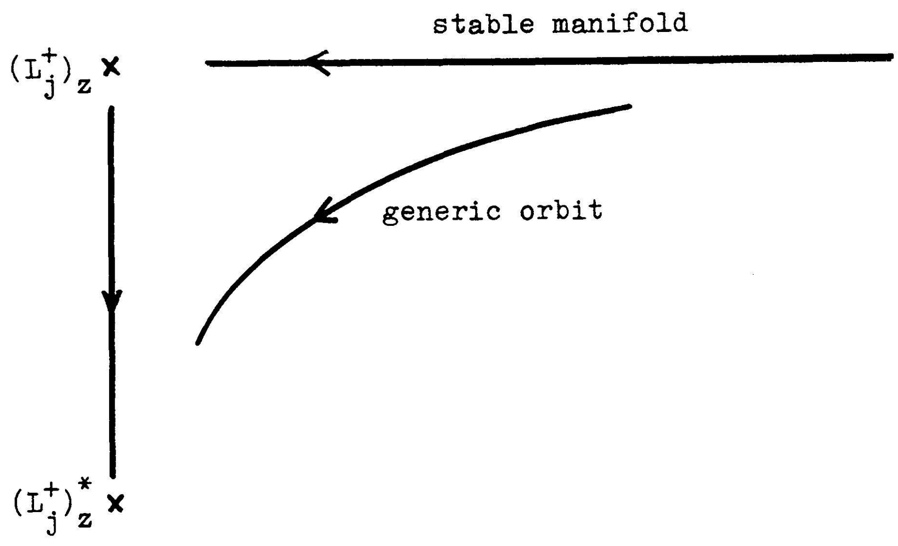

FIGURE 3.

By the stable manifold theorem with holomorphic parameter $z \in S_j^+$, we get a $\mathbf{C}^*$-invariant, $\mathrm{codim}_{\mathbf{C}}$ 1, complex submanifold $[L_j^+]$ of $P(F)$, consisting of precisely those orbits that flow into L_j^+. For the stable manifold theorem see Hadamard [16]. On N_j^- the situation is of course similar. □

In the case of monopoles on H^3 these extensions extend as bundle maps from $N_j^+ = \mathbf{CP}^3 - P_1^-$ to $\mathbf{CP}^3$ (also for N_j^-) but in our more general situations there can be obstructions to this.

The extensions of proposition 6.5 descend to the quotient $P(N_j^\pm)$, and we proceed by identifying them there. Holomorphic line bundles on the ruled surfaces are of the form:

$$\rho^* L \otimes O(n)$$

where $\rho: P(N_j^\pm) \to S_j^\pm$ is the projection, L a line bundle on $S_j^\pm$, and $O(n)$ the n-th power of the positive Hopf bundle on $P(N_j^\pm)$, which has fibre $(\mathbf{C}v)^*$ at the point $[v] \in P(N_j^\pm)$. On the fibres of $N_j^\pm$ the structure of the bundle follows from:

LEMMA 6.6. *Let* $\mathbf{C}^*$ *act on* $\mathbf{C}^2$ *by scalar multiplication. A* $\mathbf{C}^*$-*equivariant* $\mathbf{C}^2$-*bundle* $E \to \mathbf{C}^2$ *is equivariantly isomorphic to* $E_0 \times \mathbf{C}^2$ *with* E_0 *the representation of* $\mathbf{C}^*$ *on the fibre over* $0 \in \mathbf{C}^2$.

Proof (see Atiyah [2]). On $\mathbf{C}^2 \backslash \{0\}$ a $\mathbf{C}^*$-equivariant bundle is the same as a bundle on $\mathbf{CP}^1$, i.e. a sum of powers of the Hopf bundle. This establishes the given isomorphism on $\mathbf{C}^2\backslash\{0\}$. By Hartog's theorem it extends to $\mathbf{C}^2$. □

The point of the lemma is that it identifies $K_j^{\pm}$ as the pull back of $L_j^{\pm}$ under the projection $N_j^{\pm} \to S_j^{\pm}$, with $\mathbf{C}^*$ acting on it by a character of weight $\pm\, p_j$. Now one concludes readily that the extension on $P(N_j^+)$ reads:

6.3 $$0 \to \mathscr{L}_j^+ \to \mathscr{F} \to (\mathscr{L}_j^+)^* \to 0 \quad \text{with}$$
$$\mathscr{L}_j^+ = \rho^* L_j^+ \otimes O(p_j) \quad \text{and} \quad \mathscr{F} = [F_{|N_j^+\backslash\{0\}}]/\mathbf{C}^* .$$

Similarly on $P(N_j^-)$ we get:

6.4 $$0 \to \mathscr{L}_j^- \to \mathscr{F} \to (\mathscr{L}_j^-)^* \to 0 \quad \text{with}$$
$$\mathscr{L}_j^- = \rho^* L_j^- \otimes O(p_j) \quad \text{and} \quad \mathscr{F} = [F_{|N_j^-\backslash\{0\}}]/\mathbf{C}^* .$$

This results in:

THEOREM 6.7. *The monopole A defines extensions of $\mathscr{F}$ on $P(N_j^+)$ and $P(N_j^-)$ for $j = 1, \ldots, N$ as in 6.3 and 6.4. These extensions are interchanged by the real structure.* □

In the case of monopoles on H^3 these restrictions are essentially all the data one obtains about the quotient bundles and the monopole is determined by the extensions and the real structure: see Atiyah [2]. In our case the intersection of N_i^+ with N_j^- will generally have many components and we get extra data in the form of a set of invariant holomorphic identifications:

6.5 $$g_{ij}\colon N_i^+ \cap N_j^- \to \mathrm{Hom}\,(F_{|N_i^+}, F_{|N_j^-}) .$$

Conjecture. Under general conditions on the hyperbolic structure on M bundles F arising from irreducible monopoles are determined by the extensions 6.3, 6.4 and the real structure on these. □

One can almost certainly prove that if F_0 and F_1 are two holomorphic bundles on Z such that upon restriction to $\cup_i(N_i^+ \cup N_i^-)$ they become isomorphic, then they are isomorphic on Z. In order to prove the conjecture it remains to show that for irreducible monopoles no information is contained in the g_{ij}. Evidence for this conjecture comes from Thurston's version of Mostow's theorem (see Morgan [29]). This theorem implies that the flat $PSL(2, \mathbf{C})$-bundles encoding the holonomy of the hyperbolic structure are determined by their restriction to the fixed surfaces, despite the fact that the fundamental group of Z is not necessarily generated by that of the fixed surfaces. In fact one may hope to reverse this procedure: a proof of the conjecture would be a good first step towards a proof of Mostow's theorem.

It might be a good point to stress that although Z is not Kähler, suddenly algebraic objects such as elements of Picard groups and ruled surfaces have appeared. This makes algebraic geometry enter the picture, perhaps somewhat unexpectedly.

Next we shall consider spectral curves, of which we shall obtain a whole bunch instead of just a single one, as obtained in the case of $\mathbf{R}^3$ and H^3 (see Hitchin [20] and Atiyah [2]). Just as in the $\mathbf{R}^3$ and H^3 case we should compare two extensions. On $P(N_j^+ \cap N_k^-)$ we have:

6.6
$$0 \to \mathscr{L}_j^+ \to \mathscr{F} \to (\mathscr{L}_j^+)^* \to 0 \quad \text{and}$$
$$0 \to \mathscr{L}_k^- \to \mathscr{F} \to (\mathscr{L}_k^-)^* \to 0 \; .$$

Definition 6.8. The spectral curve

$$C_{jk} \subset P(N_j^+ \cap N_k^-) = (\Omega_j^+ \times \Omega_k^-)/\Gamma \qquad j, k = 1, \ldots, n$$

is the zero set of the canonical map

$$\mathscr{L}_j^+ \to (\mathscr{L}_k^-)^*$$

arising from 6.6. □

Hence for a manifold with N ends, we get N^2 spectral curves. However, the real structure clearly interchanges C_{jk} with C_{kj}, so effectively we are left with $(N^2+N)/2$ spectral curves, N of which, namely the C_{jj}, have to satisfy reality constraints. The curves can be interpreted geometrically as follows:

Proposition 6.9. *The following three are equivalent:*

1) A $\mathbf{C}^*$ *orbit* $\mathcal{O} \in (\Omega_j^+ \times \Omega_k^-)/\Gamma$ *lies in* C_{jk}.

2) The bundle F *restricted to* $\bar{\mathcal{O}} \cong P_1 \subset Z$ *is isomorphic to* $\mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$. *(For other good orbits it will be isomorphic to* $\mathcal{O}(p_j-p_k) \oplus \mathcal{O}(-p_j+p_k)$.*)*

3) The Hitchin equation (compare Hitchin [20]*):*

$$\frac{\partial s}{\partial l} + A_1 \cdot s + i\Phi \cdot s = 0 \, , \qquad s\colon g(\mathcal{O}) \to \mathbf{C}^2$$

on the corresponding geodesic $g(\mathcal{O}) \subset H^3/\Gamma$ *has a bounded solution.*

Proof. To see the equivalence of 1) and 2) we first digress on bundles on $\mathbf{CP}^1$. The result of lemma 6.6 also holds if one replaces $\mathbf{C}^2$ by $\mathbf{C}$; this follows by using an arbitrary projection $\mathbf{C}^2 \to \mathbf{C}$ and pulling back. Thus $E_{|\bar{\mathcal{O}}}$ trivializes in a $\mathbf{C}^*$-equivariant way as:

$$\begin{array}{lll} L_j^+ \oplus (L_j^+)^* & \text{on} & \bar{\mathcal{O}} - \{\infty\} \\ L_j^- \oplus (L_j^-)^* & \text{on} & \bar{\mathcal{O}} - \{0\} . \end{array}$$

The $\mathbf{C}^*$-equivariant automorphisms of $E_{|\bar{\mathcal{O}}-\{\infty\}}$ are easily seen to be of the form $\begin{bmatrix} a & b \cdot z^{2p_j} \\ 0 & c \end{bmatrix}$, and thus form a Borel subgroup of $GL(2, \mathbf{C})$. The situation is the same at infinity, and from this it follows that isomorphism classes of $\mathbf{C}^*$-equivariant holomorphic bundles on $\mathbf{CP}^1$ are given by the set of two elements $B \backslash GL(2, \mathbf{C})/B$. The exceptional case is that in which the transition function maps L_j^+ to L_j^-, i.e. $\mathcal{O} \in C_{jk}$. Then $F_{|\bar{\mathcal{O}}}$ equals $\mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$, otherwise it is isomorphic to $\mathcal{O}(p_j-p_k) \oplus \mathcal{O}(p_k-p_j)$.

To prove the equivalence of 2) and 3), we first remark that $F_{|\bar{\mathcal{O}}}$ has a bounded $\mathbf{C}^*$-invariant holomorphic nonzero section, iff $F_{|\bar{\mathcal{O}}} \cong \mathcal{O}(p_j+p_k) \oplus \mathcal{O}(-p_j-p_k)$. This follows from the standard description of sections of line bundles over $\mathbf{CP}^1$ as homogeneous polynomials and from the fact that the weights of the action are is p_j at 0 and $-p_k$ at ∞. The Hitchin equation is nothing but the Cauchy-Riemann equation for invariant sections, see Hitchin [20]. Therefore the proposition follows. □

Remark 6.10. 1) One expects that the spectral curves will generally not be compact and more or less resemble a curve of infinite genus. This is because on the universal cover H^3 we are dealing with a monopole of infinite charge.

2) It should also be remarked that the complex manifolds $(\Omega_j^+ \times \Omega_k^-)/\Gamma$ in which the spectral curves lie are far from nice generally. In the case of cyclic groups they are a $\mathbf{C}^*$-bundle over a torus and for quasi-Fuchsian groups they are disc bundles over a Riemann surface of genus $\geqslant 2$. Generally they will be Ω_j^+ bundles over S_k^- and the fibre will have infinitely many components; see § 2 where we discussed Kleinian groups.

As remarked in the introduction, it should be very interesting to find constructions for monopole bundles on these twistor spaces. It seems however that methods previously employed for $\mathbf{CP}^3$ fail, mainly due to the fact that the twistor spaces are not Kähler.