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A KALUZA-KLEIN APPROACH TO HYPERBOLIC THREE-MANIFOLDS

by Peter J. BRAAM

§ 1. INTRODUCTION

In the recent past Thurston has caused a revolution in three-dimensional topology with the creed: "Every 3-manifold is essentially geometric". In particular a large class of 3-manifolds with boundary can be supplied with a hyperbolic structure. This situation is much the same as that for two-dimensional surfaces, which can also be given hyperbolic structures.

Another even more recent revolution in mathematics came about when mathematicians started paying close attention to the methods employed in theoretical physics. In particular S. K. Donaldson found deep applications of Yang-Mills theory to four-dimensional topology.

On three-dimensional manifolds there exists a set of partial differential equations, the Bogomol'nyi equations, which describe magnetic monopoles in M. This equation is closely related to the Yang-Mills equation in dimension four, and can only be formulated in presence of a Riemannian metric and orientation on the 3-manifold. In the last three sections of this paper we shall study some aspects of this equation on hyperbolic 3-manifolds.

Kaluza-Klein theory, another favorite of the physicists, leads to a natural way to study these equations, thereby circumventing a large amount of analysis associated with more direct approaches. Basically Kaluza-Klein theory amounts to studying space through the geometry of a fibre bundle over space. In our case this fibre bundle over a hyperbolic 3-manifold is simply the product of the manifold with the circle. The analytical problems alluded to above are largely due to the fact that a 3-manifold with boundary, supplied with a hyperbolic metric, is very non-compact as a metric space. Although this is not changed by taking the product with a circle, it turns out that this 4-manifold has a natural conformal compactification (yet another popular ingredient in physical theories).

The upshot is (§ 2) that we canonically associate a conformally flat, compact 4-manifold (without boundary) with a circle action, to a hyperbolic

3-manifold (provided some conditions are satisfied see § 2). This provides a link between conformal geometry in dimension 4, and hyperbolic geometry in dimension 3. It is very similar to Poincaré's observation in 1883 that hyperbolic geometry in dimension 3 is related to conformal geometry in dimension 2, by considering the boundary surfaces of a hyperbolic 3-manifold.

In going over to the 4-manifold, no information is lost. This allows one to deduce precise facts concerning the 3-manifold from known facts about conformally flat 4-manifolds; therefore, before we start studying the Bogomol'nyi equation, we study some global differential geometric questions concerning hyperbolic 3-manifolds in the light of the conformal compactifications.

In particular we can exploit recent work of Schoen and Yau to classify a family of hyperbolic 3-manifolds (§ 3), namely those which are geometrically finite without cusps and have a limit set of Hausdorff dimension $\leqslant 1$.

On the analytical side, knowledge about conformally invariant differential operators in dimension 4 can be exploited to obtain a Hodge theory for hyperbolic 3-manifolds (§ 4). This answers a question posed by Thurston. We prove that the L^2-cohomology in dimension 1 of the 3-manifold is equal to the de Rham cohomology with compact supports. On the universal cover H^3, Poisson transformation gives an identification between closed and co-closed one forms on H^3 and closed hyperfunction one forms on δH^3. Our L^2 harmonic forms now correspond to closed, invariant currents on δH^3 with support in the limit set. Additionally this theory gives an invariant of the hyperbolic structure, of a type familiar from algebraic geometry.

After these digressions we start studying magnetic monopoles on the hyperbolic 3-manifolds by relating them to S^1-invariant instantons on the 4-manifolds. Relevant definitions and background can be found in § 5.

The twistor spaces associated to the conformally flat 4-manifolds are studied in § 6. Not only do these provide a way to study monopoles, they also encode a wealth of geometrical information belonging to the 3-manifold such as the entire geodesic flow. Finally in § 7, we use the twistor theory to construct some explicit formulas for monopoles on handlebodies. Here we naturally encounter the Eisenstein series associated to the hyperbolic 3-manifold.

We end this introduction by briefly indicating what kind of future developments can be expected. The compact 4-manifolds should allow for easy study of many natural differential operators on the 3-manifold; in § 4 it is indicated how. Using generalizations of Poisson transformation to fields of higher spin, it seems very likely that a wealth of hyperfunction objects with support in the limit set can be obtained. The twistor spaces may provide a natural environment to study theorems about the 3-manifold which rely on properties of the geodesic flow. In particular one could try to prove Mostow's theorem (and Thurston's generalisation of it) along the lines outlined in § 6.

From an analytical study of monopoles it is known that monopoles exist under reasonable conditions. This shows that there are interesting holomorphic bundles on twistor space. Understanding the structure of these will almost certainly reveal a large amount of geometry and analysis associated to the hyperbolic manifold. Finally, properties of the moduli spaces of monopoles which are independent of the metric on the 3-manifold are topological invariants of the 3-manifold. This is related to the work of Donaldson and Casson.

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§ 2. Conformal compactifications and their topology

Let $\bar{M}$ be an oriented, irreducible, atoroidal, compact, three-dimensional manifold with non-empty boundary $\delta\bar{M}$. *Atoroidal* means that every map $T^2 \to \bar{M}$ has a kernel on the level of fundamental groups. For simplicity we shall avoid cusps and thus we assume that:

2.1 either no component of $\delta\bar{M}$ is of genus 1 or $\bar{M} = \bar{D}^2 \times S^1$.

Thurston's uniformization theorem (see Morgan [29]) asserts that there is a complete, geometrically finite, hyperbolic structure on $M = \bar{M} - \delta\bar{M}$. This means that M can be realised as follows (see Bers [7], Maskit [27], Morgan [29], Beardon [6] for background).

Recall that $PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\{\pm 1\}$ is the isometry group of hyperbolic 3-space H^3, and that the right action of an isometry on $H^3 = SU(2)\backslash SL(2, \mathbf{C})$ extends over the boundary $S^2 \cong \delta H^3$ as an action by a fractional linear transformation of S^2. A *Kleinian group* Γ without cusps is a discrete