

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	34 (1988)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	 AN ELEMENTARY PROOF OF THE STRUCTURE THEOREM FOR CONNECTED SOLVABLE AFFINE ALGEBRAIC GROUPS
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DOI:	https://doi.org/10.5169/seals-56599

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AN ELEMENTARY PROOF OF THE STRUCTURE THEOREM FOR CONNECTED SOLVABLE AFFINE ALGEBRAIC GROUPS

by Dragomir Ž. ĐOKOVIĆ¹⁾

ABSTRACT

We give an elementary proof of the basic structure theorem for connected solvable affine algebraic groups G over an algebraically closed field k . The main feature of our proof is that we first establish the important fact that the centralizer in G of a semisimple element s is connected. Then the main structure theorem follows easily. We also prove that such s is contained in a maximal torus and that all maximal tori of G are conjugate. The structure theorem for connected nilpotent affine groups is not needed in the proof; it is obtained at the end as a simple consequence of the main results. In our proof we avoid the use of quotients and Lie algebras of affine groups. On the other hand we use the Lie-Kolchin theorem, Chevalley's theorem, the existence and uniqueness of the Jordan decomposition, and some other elementary facts.

Let k be an algebraically closed field. All algebraic groups will be defined over k and are assumed to be affine. By $N \rtimes H$ we denote the semidirect product of affine algebraic groups where N is a normal and H a complementary subgroup. If G is any affine algebraic group we shall denote by G_u (resp. G_s) the set of all unipotent (resp. semisimple) elements of G . By G^0 we denote the identity component of G and by G' the derived subgroup of G . A torus S in G will be called maximal if $S \subset T$ implies that $T = S$ for any torus T of G . The center of G is denoted by $Z(G)$. The centralizer of $s \in G$ resp. $S \subset G$ in a subgroup $H \subset G$ will be denoted by $Z_H(s)$ resp. $Z_H(S)$. The existence and uniqueness of the Jordan decomposition for elements of G will be used without explicit reference. All group homomorphisms will be homomorphisms of affine algebraic groups. For other proofs of the structure theorem for connected solvable affine algebraic groups we refer the reader to the references [1-5].

¹⁾ This work was supported by the NSERC of Canada Grant A-5285.

1980 *Mathematics Subject Classification*: primary 20G15, secondary 22E25.

The group of all invertible $n \times n$ upper triangular matrices will be denoted by B_n . Its subgroup consisting of all diagonal matrices is denoted by D_n . We have $B_n = U_n \rtimes D_n$ where U_n is the closed connected subgroup of B_n consisting of all unipotent elements of B_n .

We start with some preliminary facts.

THEOREM 1 (Lie-Kolchin). *Every connected solvable affine algebraic group can be embedded in some B_n as a closed subgroup.* \square

COROLLARY. *If G is a connected solvable affine group then $G' \subset G_u$.* \square

THEOREM 2 (Chevalley). *If N is a closed normal subgroup of an affine group G then there exists a homomorphism $f: G \rightarrow GL_n(k)$ such that $\text{Ker } f = N$.* \square

For the proofs of Theorems 1 and 2 see, for instance, [5, Theorems 6.7 and 5.1.3].

LEMMA 1. *If $f: G \rightarrow H$ is a surjective homomorphism of affine algebraic groups and $N := \text{Ker } f$ then:*

- (i) $f(G^0) = H^0$;
- (ii) $f(G_u) = H_u$ and $f(G_s) = H_s$;
- (iii) $\dim G = \dim N + \dim H$;
- (iv) *If N and H are connected then G is connected.*

Proof. For the proofs of (i) and (iii) see for instance [4, Section 7.4]. (ii) follows from the fact that f preserves the Jordan decomposition [4, Theorem 2.4.8]. We shall sketch the proof of (iv). Since N is connected, we have $N \subset G^0$. By (i) we have $f(G^0) = H^0 = H$, and consequently $G = NG^0 = G^0$. \square

We need a lemma to prove the centralizer theorem. For a more general version of this lemma see [2, Proposition (9.3)].

LEMMA 2. *Let N be a closed normal connected abelian unipotent subgroup of an affine group G and let $s \in G_s$. Then $M := \{sus^{-1}u^{-1} : u \in N\}$ is a closed connected subgroup of N , the multiplication map $\mu: M \times Z_N(s) \rightarrow N$ is bijective, and $Z_N(s)$ is connected.*

Proof. Since N is abelian, the map $f: N \rightarrow N$, defined by $f(u) = sus^{-1}u^{-1}$, is a morphism of algebraic groups whose kernel is $Z_N(s)$ and image M , so M is a closed connected subgroup of N . If $x \in M \cap Z_N(s)$ then $x = sus^{-1}u^{-1}$ for some $u \in N$. Since $usu^{-1} = x^{-1}s = sx^{-1}$ is semisimple and x is unipotent, the uniqueness of the Jordan decomposition implies that $x = 1$. Hence $M \cap Z_N(s) = 1$ and so μ is injective. By Lemma 1 (iii) we have $\dim N = \dim M + \dim Z_N(s)$, which implies that the homomorphism μ is also surjective, i.e., $MZ_N(s) = N$. The same argument shows that $MZ_N(s)^0 = N$, and so $Z_N(s)$ must be connected. \square

THEOREM 3. *If G is a connected solvable affine group and $s \in G_s$ then $Z_G(s)$ is connected and $G = G_u Z_G(s)$.*

Proof. We use induction on $\dim G$. If G is abelian the assertions are trivial. Otherwise let N be the last non-trivial term of the derived series of G . By the Corollary of Theorem 1, N is unipotent. We now apply Theorem 2 to this G and N . Let f be as in that theorem. We shall write \bar{x} for $f(x)$ and \bar{G} for $f(G)$.

Let $z \in G$ be such that $\bar{z} \in Z_{\bar{G}}(\bar{s})$. Then $szs^{-1}z^{-1} \in N$. By Lemma 2 there exists $u \in N$ and $v \in Z_N(s)$ such that $szs^{-1}z^{-1} = sus^{-1}u^{-1} \cdot v$. Since v commutes with u and s , and $zsz^{-1} = v^{-1} \cdot usu^{-1}$, it follows that $v = 1$. Thus $u^{-1}z \in Z_G(s)$ and consequently we have a short exact sequence

$$1 \rightarrow Z_N(s) \hookrightarrow Z_G(s) \rightarrow Z_{\bar{G}}(\bar{s}) \rightarrow 1.$$

By Lemma 2, $Z_N(s)$ is connected. By Lemma 1 (iii) we have $\dim \bar{G} < \dim G$. By induction hypothesis, we conclude that $Z_{\bar{G}}(\bar{s})$ is connected and that $\bar{G} = (\bar{G})_u \cdot Z_{\bar{G}}(\bar{s})$. Now Lemma 1 (iv) implies that $Z_G(s)$ is connected. By part (ii) of the same lemma we have $f(G_u) = (\bar{G})_u$ and so $f(G_u Z_G(s)) = (\bar{G})_u Z_{\bar{G}}(\bar{s}) = \bar{G}$. Since $N \subset G_u$, it follows that $G = G_u Z_G(s)$. \square

We now proceed to prove the main results about the structure of connected solvable affine groups. But first we need two lemmas.

LEMMA 3. *Let $S \subset B_n$ be a commuting set of semisimple elements. Then there exists $b \in B_n$ such that $b^{-1}Sb \subset D_n$.*

Proof. It is an elementary fact of linear algebra that there exists $a \in GL_n(k)$ such that $a^{-1}Sa \subset D_n$. Hence if $M_n(k)$ is the algebra of n by n matrices over k and A its subalgebra generated by S , we know that A is semisimple (and commutative). Let $V := k^n$ be the space of column

vectors and let e_1, \dots, e_n be its standard basis. We shall view V as a left $M_n(k)$ -module via matrix multiplication. The subspace V_i spanned by the vectors e_1, \dots, e_i is an A -submodule of V for each i . Since A is semisimple, there exist $v_i \in V_i \setminus V_{i-1}$, $1 \leq i \leq n$, such that $Av_i = kv_i$. Thus if b is the matrix whose i -th column is v_i , $1 \leq i \leq n$, then $b \in B_n$ and $b^{-1}Sb \subset D_n$. \square

LEMMA 4. *If G is a connected solvable affine group, $T \subset G_s$ a closed subgroup, and $G = G_u T$ then T is a torus and $G = G_u \rtimes T$.*

Proof. By the Lie-Kolchin theorem we may assume that G is a closed subgroup of some B_n . By using the projection map $B_n \rightarrow D_n$ we obtain a short exact sequence $1 \rightarrow G_u \hookrightarrow G \xrightarrow{p} D \rightarrow 1$, where $D \subset D_n$ is a torus. Since $D = p(G) = p(G_u T) = p(T)$, Lemma 1 (i) implies that $p(T^0) = D$. Thus $G = G_u T^0$ and using $T \cap G_u = 1$ we conclude that $T = T^0$. In particular T is abelian and by Lemma 3 we may assume that $T \subset D_n$, i.e., $T = D$. Since $B_n = U_n \rtimes D_n$, $G_u \subset U_n$, $T = D \subset D_n$, and $G = G_u T$, it follows that $G = G_u \rtimes T$. \square

THEOREM 4. *Let G be a connected solvable affine group. Then $G = G_u \rtimes T$ where T is a maximal torus. In particular, G_u is connected.*

Proof. We use induction on $\dim G$. Assume first that $G_s \subset Z(G)$. Then $G_s = Z(G)_s$ is a closed subgroup of G and $G = G_u G_s$. The assertion then follows from Lemma 4. Now assume that there exists $s \in G_s \setminus Z(G)$. Then $Z_G(s)$ is a proper closed subgroup of G , see e.g. [4, Section 8.2]. By Theorem 3 it is connected and $G = G_u Z_G(s)$. By induction hypothesis there exists a torus T such that $Z_G(s) = Z_G(s)_u T$. Then $G = G_u Z_G(s) = G_u T$ and $G = G_u \rtimes T$ by Lemma 4. \square

THEOREM 5. *Let $G = G_u \rtimes T$ be a connected solvable affine group. Then every $s \in G_s$ is conjugate to an element of T .*

Proof. We use induction on $\dim G$. We have $s = ut$ where $u \in G_u$ and $t \in T$. If G is abelian then $u = 1$ and $s = t$. Otherwise let N be the last non-trivial term of the derived series of G . By the corollary of Theorem 1 we have $N \subset G_u$. Hence N is a closed connected normal abelian unipotent subgroup of G . By Theorem 2 and the induction hypothesis there exists $x \in G$ such that $x s x^{-1} = t v$ where $v \in N$. By Lemma 2, $v = t^{-1} u t u^{-1} z$ where $u \in N$ and $z \in Z_N(t)$. Hence $x s x^{-1} = u t u^{-1} z$. Since $x s x^{-1}, u t u^{-1} \in G_s$, $z \in G_u$, and z commutes with u and t , it follows that $z = 1$ and consequently $x s x^{-1} = u t u^{-1}$. \square

THEOREM 6. *If $G = G_u \times T$ is a connected solvable affine group and $S \subset G_s$ is a commuting set then $Z_G(S)$ is connected and $aSa^{-1} \subset T$ for some $a \in G$. In particular, all maximal tori of G are conjugate.*

Proof. We use induction on $\dim G$. The assertions are obvious if $S \subset Z(G)$. Otherwise choose $s \in S \setminus Z(G)$. By Theorem 5 we may assume that $s \in T$. Then $Z_G(s)$ is a proper closed subgroup of G containing T and S . By Theorem 3, $Z_G(s)$ is connected. Since $\dim Z_G(s) < \dim G$, we can apply the induction hypothesis to conclude the proof. \square

It is now easy to describe connected nilpotent affine groups.

THEOREM 7. *A connected solvable affine group $G = G_u \times T$ is nilpotent if and only if $G_s = T \subset Z(G)$. In that case $G = G_u \times T$.*

Proof. Assume that G is nilpotent. We prove that $G_s = T \subset Z(G)$ by induction on $\dim G$. We may assume that G is not abelian. Let N be the last non-trivial term of the lower central series of G . Let f be as in Theorem 2 and $\bar{G} = f(G)$. Then $\bar{G} = f(G_u T) = (\bar{G})_u f(T)$. By induction hypothesis we have $f(T) = (\bar{G})_s \subset Z(\bar{G})$. Consequently if $t \in T$ and $x \in G$ then $u := txt^{-1}x^{-1} \in N$. Since $N \subset Z(G) \cap G_u$, and $xtx^{-1} = u^{-1}t = tu^{-1}$ we must have $u = 1$. Thus $T \subset Z(G)$ and, by Theorem 5, $G_s = T$. The converse is obvious. \square

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(Reçu le 23 janvier 1988)

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