

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 34 (1988)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS
Autor: Piveteau, Jean-Marc
Kapitel: 2. Orthogonal representations of p-groups
DOI: <https://doi.org/10.5169/seals-56597>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 03.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

We can identify the irreducible faithful $\mathbf{Q}[\mathbf{Z}/p]$ -Module \mathbf{Q}^{p-1} with $\mathbf{Q}(\zeta_p)$ (ζ_p : primitive p -th root of unity, $1 \in \mathbf{Z}/p$ acts on $\mathbf{Q}(\zeta_p)$ by multiplication with ζ_p). Any symmetric σ -invariant bilinear form is given by $tr_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(axy\bar{y})$ with $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ (cf. [4] or [6]). We write γ_a for the σ -invariant bilinear form corresponding to $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$.

(1.2) LEMMA. *The discriminant of γ_a in $\mathbf{Q}/\mathbf{Q}^{*2}$ is equal to $p \bmod \mathbf{Q}^{*2}$.*

Proof. Since $a \in \mathbf{L} := \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ we have: $\gamma_a = tr_{\mathbf{L}/\mathbf{Q}}(tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}axy\bar{y})$. An easy computation shows that $tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}(axy\bar{y})$ is a 2-dimensional symmetric \mathbf{L} -bilinearform with discriminant $4 - (\zeta_p + \zeta_p^{-1})^2 \bmod \mathbf{L}^{*2} \in \mathbf{L}/\mathbf{L}^{*2}$. Applying [7, Lemma 2.2] we conclude that the discriminant of γ_a is independant of $a \in \mathbf{L}$. Consider now the matrix representation of σ given before (σ : irreducible faithful \mathbf{Q} -representation of \mathbf{Z}/p). Let C be the $(p-1) \times (p-1)$ -matrix given by:

$$C := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \cdot & \cdot \\ & & & -1 & 2 \end{bmatrix}$$

It is easy to check that C is the matrix of a σ -invariant symmetric bilinear form. The Lemma follows since the determinant of C is equal to p .

2. ORTHOGONAL REPRESENTATIONS OF p -GROUPS

Let $p > 2$ be an odd prime. The integer $l_{\mathbf{Q}}(p)$ is defined by

$$l_{\mathbf{Q}}(p) := \text{g.c.d.} \left\{ \begin{array}{l} m > 1 \\ \text{the } m\text{-fold direct sum } \sigma \oplus \dots \oplus \sigma \text{ of the irreducible faithful } \mathbf{Q}\text{-representation } \sigma \text{ of } \mathbf{Z}/p \text{ is equivalent to an orthogonal representation} \end{array} \right\}$$

The importance played by cyclic groups in the investigation of representations of p -groups is given by the following result (cf. [1, Theorem (1.10)]):

(2.1) PROPOSITION. *Let G be a finite p -group ($p > 2$) and let ρ be an irreducible \mathbf{Q} -representation of G . Then either ρ is induced from a representation θ of a normal subgroup of index p , or ρ factors through a \mathbf{Q} -representation of \mathbf{Z}/p .*

The degree of an irreducible non trivial \mathbf{Q} -representation of a finite p -group is therefore of the form $p^k(p-1)$ ($k=0, 1, 2, \dots$), cf. [1, Corollary (1.11)].

(2.2) PROPOSITION. Let G be a p -group ($p > 2$) and $\rho: G \rightarrow SO_{2m}(\mathbf{Q})$ a representation of G with $2m \not\equiv 0 \pmod{l_{\mathbf{Q}}(p) \cdot (p-1)}$. Then ρ has a fixed point (i.e. $\rho = 1 \oplus \tau$ where 1 is the unique 1-dimensional \mathbf{Q} -representation of G).

We will need the following lemma for the proof of (2.2):

(2.3) LEMMA. Let $\rho: G \rightarrow GL_m(\mathbf{Q})$ be an irreducible non trivial representation of the p -group G ($p > 2$) and let ψ be a ρ -invariant symmetric bilinear form. If we write σ for the irreducible faithful representation of \mathbf{Z}/p , then there exist σ -invariant bilinear forms $\Gamma_1, \dots, \Gamma_s$ such that ψ is equivalent to the orthogonal sum $\Gamma_1 \perp \dots \perp \Gamma_s$.

Proof. Let $p^k(p-1)$ be the degree of ρ . We prove the lemma by induction on k . For $k = 0$, ρ factors through the irreducible faithful representation σ of \mathbf{Z}/p . Every ρ -invariant symmetric bilinear form ψ is therefore σ -invariant. For $k > 0$, ρ is induced by a representation θ of a normal subgroup H of index p . The restriction ρ_H of ρ to H splits in a direct sum: $\rho = \theta_1 \oplus \dots \oplus \theta_p$ with $\theta = \theta_1$ and θ_i is irreducible for $i = 1, \dots, p$. By (1.1) we can assume that \mathbf{Q}^m is the orthogonal sum of the corresponding irreducible invariant subspaces. The assertion follows by induction.

Proof of (2.2). If $G = \mathbf{Z}/p$, we split ρ in a direct sum: $\rho = n_0 1 \oplus n_1 \sigma$ (1 : one dimensional representation of \mathbf{Z}/p ; σ : irreducible faithful representation of \mathbf{Z}/p). If $n_0 = 0$ then n_1 must be a multiple of $l_{\mathbf{Q}}(p)$, i.e. we have $2m \equiv 0 \pmod{(p-1)l_{\mathbf{Q}}(p)}$. Contradiction.

If G is not \mathbf{Z}/p , we split ρ in a direct sum of irreducible representations: $\rho = \rho_1 \oplus \dots \oplus \rho_t$, chosen in such a way that \mathbf{Q}^{2m} is the orthogonal sum of the corresponding invariant subspaces. Suppose now that ρ has no fixed points. Then all ρ_i are non trivial and it follows from (2.3) that any ρ -invariant symmetric bilinear form is equivalent to an orthogonal sum of σ -invariant symmetric bilinear forms. We can therefore construct a representation $\mathbf{Z}/p \rightarrow SO_{2m}(\mathbf{Q})$ without fixed points, what contradicts the first part of the proof.

The rest of the section is devoted to the computation of $l_{\mathbf{Q}}(p)$, p odd prime.

(2.4) PROPOSITION. $l_{\mathbf{Q}}(p) = \begin{cases} 2 & \text{if } p \not\equiv 7 \pmod{8} \\ 4 & \text{otherwise.} \end{cases}$

Proof. For each $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$, the discriminant of γ_a is not a square in \mathbf{Q} (cf. lemma (1.2)). Therefore $l_{\mathbf{Q}}(p)$ must be even. The 4-fold orthogonal sum of a \mathbf{Q} -bilinear form is equivalent to the standard bilinear form, since every integer is sum of four squares. Let C be the matrix considered in the proof of lemma (1.2). If it is possible to find two rational numbers u and v such that the matrix $X_{u,v}$

$$X_{u,v} := \begin{bmatrix} uC & 0 \\ 0 & vC \end{bmatrix}$$

represents a bilinear form $\xi_{u,v}$ which is equivalent to the standard one, then the representation $\sigma \oplus \sigma$ is equivalent to an orthogonal representation. This sufficient condition is also necessary if $p = 3 \pmod 4$ (cf. [5]). For a prime p , let \mathbf{Q}_p be the field of p -adic numbers and write \mathbf{Q}_∞ for \mathbf{R} as usual. For $a, b \in \mathbf{Q}$ and for $v = 2, 3, 5, 7, \dots, \infty$ we write $(a, b)_v$ for the Hilbert symbol of a and b relatively to \mathbf{Q}_v . For a bilinearform α given in an orthogonal base by the diagonal matrix

$$\begin{bmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{bmatrix}$$

we write $H_v(\alpha)$ ($v = 2, 3, \dots, \infty$) for the Hasse invariant, which is defined by

$$H_v(\alpha) = \prod_{i < j} (a_i, a_j)_v$$

Using the formulas given for example by [9] to compute the Hilbert symbol, one check that:

$$\begin{aligned} H_v(\xi_{1,1}) &= 1 && \text{if } p \neq 3 \pmod 4 && \text{for } v = 2, 3, 5, 7, \dots, \infty, \\ H_2(\xi_{u,v}) &= -1 && \text{if } p = 7 \pmod 8 && \text{for any } u \text{ and any } v, \\ H_v(\xi_{2p,1}) &= 1 && \text{if } p = 1 \pmod 8 && \text{for } v = 2, 3, 5, 7, \dots, \infty. \end{aligned}$$

Since the discriminant of $\xi_{u,v}$ is $1 \in \mathbf{Q}/\mathbf{Q}^{*2}$ and since $\xi_{u,v}$ is positive definit for any u and any v , it follows that $\sigma \oplus \sigma$ is equivalent to an orthogonal representation if and only if $p \neq 7 \pmod 8$. It remains to show that, for $p = 7 \pmod 8$, the $2n$ -fold orthogonal sum μ given by the matrix H :

$$H := \begin{bmatrix} u_1 C & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & u_{2n} C \end{bmatrix}$$

is isomorphic to the standard bilinear form if and only if n is even. Let u_{odd} and u_{even} defined by:

$$u_{\text{even}} := \prod_{k=1}^n u_{2k} \quad u_{\text{odd}} := \prod_{k=1}^n u_{2k-1};$$

an easy computation shows that $H_v(\xi_{u_{\text{even}}, u_{\text{odd}}}) = H_v(\mu)$ if n is odd. The proposition follows.

3. PROOF OF THE MAIN THEOREM

(3.1) LEMMA. Let p be a prime number ($p > 2$). For every integer m satisfying $2m \not\equiv 0 \pmod{(p-1) \cdot l_{\mathbf{Q}}(p)}$ we have $F_{\mathbf{Q}}(m, p) = 1$.

Proof. Let G be a p -group, $p > 2$. It follows from (2.2) that any representation ρ of G splits: $\rho = 1 \oplus \tau$ (1 is the 1-dimensional representation of G). Then we have $e(\rho) = e(1)e(\tau) = 0$.

We are now able to prove the main theorem. It has been showed in [3] that $F_{\mathbf{Q}}(n) = 4$ if n is odd. If n is even, four cases have to be distinguished. If $p = 2$ then the $n/2^{N-2}$ -fold sum of the irreducible faithful representation of $\mathbf{Z}/2^N$, where 2^N is the 2-primary part of $\text{den}(B_n/n)$, is an orthogonal representation with Euler class of order 2^N (cf. [1]). Let now p be an odd prime. Since the irreducible faithful representation ν of \mathbf{Z}/p^r ($r \geq 1$) is induced by the irreducible faithful representation of $\mathbf{Z}/p \subset \mathbf{Z}/p^r$, the M -fold sum of ν is equivalent to an orthogonal representation if and only if $l_{\mathbf{Q}}(p)$ divides M . Write $n = Np^k(p-1)$ with $\text{g.c.d.}(N, p) = 1$. If N is even, the $2N$ -fold sum of the irreducible faithful representation of \mathbf{Z}/p^{k+1} is orthogonal and has Euler class of order p^{k+1} (cf. [1]); if N is odd and $p \not\equiv 7 \pmod{8}$ then the $2N$ -fold sum of the irreducible faithful representation of \mathbf{Z}/p^{k+1} is orthogonal and has Euler class of order p^{k+1} (cf. [1]). In the three cases, the statement follows from the well known characterization of $\text{den}(B_n/n)$ (cf. [1] for example). Eventually, applying (3.1) we see that $F_{\mathbf{Q}}(n, p) = 1$ if N is odd and $p \equiv 7 \pmod{8}$.