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## THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS

by Jean-Marc PIVETEAU

If  $\rho: G \rightarrow SO_n(\mathbf{R})$  is an orthogonal representation of the group  $G$ , then the Euler class  $e(\rho)$  is defined as Euler class of the flat real vector bundle over  $BG$  associated with  $\rho$ . For representations of finite groups over a number field  $\mathbf{K}$  there is a uniform bound, depending on  $\mathbf{K}$  and on the degree of the representation only, for the order of the Euler class. This bound has been extensively studied by Eckmann and Mislin ([1], [2], [3]). In this note we discuss analogous bounds for orthogonal representations over the field  $\mathbf{Q}$  of rational numbers. Since the best upper bound for odd dimensional representations is equal to two (cf. [3]), we consider the case of even dimensional  $\mathbf{Q}$ -representations. We will write  $F_{\mathbf{Q}}(m)$  for the best upper bound for the order of the Euler Class  $e(\rho)$ , where  $\rho$  ranges over all  $2m$ -dimensional representations of finite groups over  $\mathbf{Q}$ . Thus, for every representation  $\rho: G \rightarrow SO_{2m}(\mathbf{Q})$  of any finite group  $G$ , it follows that  $F_{\mathbf{Q}}(m) \cdot e(\rho) = 0 \in H^{2m}(G; \mathbf{Z})$ , and  $F_{\mathbf{Q}}(m)$  is the best possible. The prime factorisation of the numbers  $F_{\mathbf{Q}}(m)$  is given as follows:

MAIN THEOREM. For odd  $m$  we have  $F_{\mathbf{Q}}(m) = 4$ . For even  $m$ , if we write  $F_{\mathbf{Q}}(m, p)$  for the  $p$ -primary part of  $F_{\mathbf{Q}}(m)$  ( $p$ : prime), we have:

$$F_{\mathbf{Q}}(m, p) = \begin{cases} 1, & \text{if } n \not\equiv 0 \pmod{p-1} \text{ or if } n = Np^k(p-1) \text{ with} \\ & \text{g.c.d.}(p, N) = 1, \text{ } N \text{ odd and } p \equiv 7 \pmod{8}, \\ p\text{-primary part of } \text{den}(B_m/m) & \text{otherwise,} \end{cases}$$

where  $B_m$  is the  $m$ -th Bernoulli-number and  $\text{den}(B_m/m)$  is the denominator of  $B_m/m$  written in its lowest terms.

Note that  $F_{\mathbf{Q}}(m)$  is a lower bound for the order of the universal profinite Euler class  $\hat{e}_{2m}(\mathbf{Q})$  considered by Eckmann and Mislin in [3].

The two first sections contain preliminary results about bilinear forms and orthogonal representations. In the last section, we prove the main theorem.

This paper is a summary of some results of the thesis [8] I have written under the direction of Guido Mislin. I want to express him on this

occasion my gratefulness for his stimulating advices and the interest he constantly showed for this work.

# 1. INVARIANT BILINEAR FORMS

Let  $\mathbf{K}$  be a field of characteristic 0,  $V$  a finite dimensional vector space over  $\mathbf{K}$  and  $\rho: G \rightarrow GL(V)$  a  $\mathbf{K}$ -representation of the group  $G$ . A  $\mathbf{K}$ -bilinearform  $\alpha: V \times V \rightarrow \mathbf{K}$  is called  $\rho$ -invariant if

$$\alpha(\rho(g)x, \rho(g)y) = \alpha(x, y) \quad \forall x, y \in V, \quad \forall g \in G.$$

If  $G$  is finite, then for any bilinear form  $\gamma$  the form  $\bar{\gamma}$  defined by

$$\bar{\gamma}(x, y) := \sum_{g \in G} \gamma(\rho(g)x, \rho(g)y)$$

is  $\rho$ -invariant.

(1.1) *Remark.* If  $\alpha$  is definit (i.e.  $\alpha(x, x) = 0 \Rightarrow x = 0$ ) and if  $\rho$  splits in a direct sum  $\rho = \rho_1 \oplus \rho_2$ , the restriction  $\rho'$  of  $\rho$  to the orthogonal complement of the invariant space corresponding to  $\rho_1$  is equivalent to  $\rho_2$ . Since we always can substitute a representation or a bilinear form by an equivalent one, we can assume that the representation space of a sum is an orthogonal sum of corresponding invariant subspaces.

We call *standard bilinear form* (of dimension  $m$ ) the map  $\beta_m: \mathbf{K}^m \times \mathbf{K}^m \rightarrow \mathbf{K}$  given by

$$\beta_m(x, y) := \sum_{i=1}^m x_i y_i \quad \text{with} \quad x = (x_1, \dots, x_m) \quad \text{and} \quad y = (y_1, \dots, y_m).$$

The group  $O_m(\mathbf{K})$  is the subgroup of  $GL_m(\mathbf{K})$  of matrices  $(a_{ij})$  such that  $\sum_k a_{ik} a_{jk} = \delta_{ij}$  for all  $i, j$ . The group  $SO_m(\mathbf{K})$  is the subgroup of  $O_m(\mathbf{K})$  of matrices  $(a_{ij})$  with  $\det(a_{ij}) = 1$ . It is therefore evident that a representation  $\rho: G \rightarrow GL_m(\mathbf{K})$  is realizable over  $O_m(\mathbf{K})$  if and only if there is a  $\rho$ -invariant symmetric bilinear form which is equivalent to the standard bilinear form.

Let  $p$  be a prime number. Up to equivalence, there is a unique irreducible faithful  $\mathbf{Q}$ -representation  $\sigma$  of  $\mathbf{Z}/p$ ; it is given by

$$\sigma: \mathbf{Z}/p \rightarrow GL_{p-1}(\mathbf{Q})$$

$$1 \mapsto A := \begin{bmatrix} 0 & . & . & . & -1 \\ 1 & . & . & . & -1 \\ & & & . & \\ . & . & . & 1 & -1 \end{bmatrix}$$

We can identify the irreducible faithful  $\mathbf{Q}[\mathbf{Z}/p]$ -Module  $\mathbf{Q}^{p-1}$  with  $\mathbf{Q}(\zeta_p)$  ( $\zeta_p$ : primitive  $p$ -th root of unity,  $1 \in \mathbf{Z}/p$  acts on  $\mathbf{Q}(\zeta_p)$  by multiplication with  $\zeta_p$ ). Any symmetric  $\sigma$ -invariant bilinear form is given by  $\text{tr}_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(axy\bar{y})$  with  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$  (cf. [4] or [6]). We write  $\gamma_a$  for the  $\sigma$ -invariant bilinear form corresponding to  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ .

(1.2) LEMMA. *The discriminant of  $\gamma_a$  in  $\mathbf{Q}/\mathbf{Q}^{*2}$  is equal to  $p \bmod \mathbf{Q}^{*2}$ .*

*Proof.* Since  $a \in \mathbf{L} := \mathbf{Q}(\zeta_p + \zeta_p^{-1})$  we have:  $\gamma_a = \text{tr}_{\mathbf{L}/\mathbf{Q}}(\text{tr}_{\mathbf{Q}(\zeta_p)/\mathbf{L}}axy\bar{y})$ . An easy computation shows that  $\text{tr}_{\mathbf{Q}(\zeta_p)/\mathbf{L}}(axy\bar{y})$  is a 2-dimensional symmetric  $\mathbf{L}$ -bilinearform with discriminant  $4 - (\zeta_p + \zeta_p^{-1})^2 \bmod \mathbf{L}^{*2} \in \mathbf{L}/\mathbf{L}^{*2}$ . Applying [7, Lemma 2.2] we conclude that the discriminant of  $\gamma_a$  is independant of  $a \in \mathbf{L}$ . Consider now the matrix representation of  $\sigma$  given before ( $\sigma$ : irreducible faithful  $\mathbf{Q}$ -representation of  $\mathbf{Z}/p$ ). Let  $C$  be the  $(p-1) \times (p-1)$ -matrix given by:

$$C := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \ddots \\ & & & -1 & 2 \end{bmatrix}$$

It is easy to check that  $C$  is the matrix of a  $\sigma$ -invariant symmetric bilinear form. The Lemma follows since the determinant of  $C$  is equal to  $p$ .

## 2. ORTHOGONAL REPRESENTATIONS OF $p$ -GROUPS

Let  $p > 2$  be an odd prime. The integer  $l_{\mathbf{Q}}(p)$  is defined by

$$l_{\mathbf{Q}}(p) := \text{g.c.d.} \left\{ \begin{array}{l} m > 1 \\ \text{the } m\text{-fold direct sum } \sigma \oplus \dots \oplus \sigma \text{ of the irreducible faithful } \mathbf{Q}\text{-representation } \sigma \text{ of } \mathbf{Z}/p \text{ is equivalent to an orthogonal representation} \end{array} \right\}$$

The importance played by cyclic groups in the investigation of representations of  $p$ -groups is given by the following result (cf. [1, Theorem (1.10)]):

(2.1) PROPOSITION. *Let  $G$  be a finite  $p$ -group ( $p > 2$ ) and let  $\rho$  be an irreducible  $\mathbf{Q}$ -representation of  $G$ . Then either  $\rho$  is induced from a representation  $\theta$  of a normal subgroup of index  $p$ , or  $\rho$  factors through a  $\mathbf{Q}$ -representation of  $\mathbf{Z}/p$ .*

The degree of an irreducible non trivial  $\mathbf{Q}$ -representation of a finite  $p$ -group is therefore of the form  $p^k(p-1)$  ( $k=0, 1, 2, \dots$ ), cf. [1, Corollary (1.11)].

(2.2) PROPOSITION. Let  $G$  be a  $p$ -group ( $p>2$ ) and  $\rho: G \rightarrow SO_{2m}(\mathbf{Q})$  a representation of  $G$  with  $2m \not\equiv 0 \pmod{l_{\mathbf{Q}}(p) \cdot (p-1)}$ . Then  $\rho$  has a fixed point (i.e.  $\rho = 1 \oplus \tau$  where  $1$  is the unique 1-dimensional  $\mathbf{Q}$ -representation of  $G$ ).

We will need the following lemma for the proof of (2.2):

(2.3) LEMMA. Let  $\rho: G \rightarrow GL_m(\mathbf{Q})$  be an irreducible non trivial representation of the  $p$ -group  $G$  ( $p>2$ ) and let  $\psi$  be a  $\rho$ -invariant symmetric bilinear form. If we write  $\sigma$  for the irreducible faithful representation of  $\mathbf{Z}/p$ , then there exist  $\sigma$ -invariant bilinear forms  $\Gamma_1, \dots, \Gamma_s$  such that  $\psi$  is equivalent to the orthogonal sum  $\Gamma_1 \perp \dots \perp \Gamma_s$ .

*Proof.* Let  $p^k(p-1)$  be the degree of  $\rho$ . We prove the lemma by induction on  $k$ . For  $k=0$ ,  $\rho$  factors through the irreducible faithful representation  $\sigma$  of  $\mathbf{Z}/p$ . Every  $\rho$ -invariant symmetric bilinear form  $\psi$  is therefore  $\sigma$ -invariant. For  $k>0$ ,  $\rho$  is induced by a representation  $\theta$  of a normal subgroup  $H$  of index  $p$ . The restriction  $\rho_H$  of  $\rho$  to  $H$  splits in a direct sum:  $\rho = \theta_1 \oplus \dots \oplus \theta_p$  with  $\theta = \theta_1$  and  $\theta_i$  is irreducible for  $i=1, \dots, p$ . By (1.1) we can assume that  $\mathbf{Q}^m$  is the orthogonal sum of the corresponding irreducible invariant subspaces. The assertion follows by induction.

*Proof of (2.2).* If  $G = \mathbf{Z}/p$ , we split  $\rho$  in a direct sum:  $\rho = n_0 1 \oplus n_1 \sigma$  ( $1$ : one dimensional representation of  $\mathbf{Z}/p$ ;  $\sigma$ : irreducible faithful representation of  $\mathbf{Z}/p$ ). If  $n_0 = 0$  then  $n_1$  must be a multiple of  $l_{\mathbf{Q}}(p)$ , i.e. we have  $2m \equiv 0 \pmod{(p-1)l_{\mathbf{Q}}(p)}$ . Contradiction.

If  $G$  is not  $\mathbf{Z}/p$ , we split  $\rho$  in a direct sum of irreducible representations:  $\rho = \rho_1 \oplus \dots \oplus \rho_t$ , chosen in such a way that  $\mathbf{Q}^{2m}$  is the orthogonal sum of the corresponding invariant subspaces. Suppose now that  $\rho$  has no fixed points. Then all  $\rho_i$  are non trivial and it follows from (2.3) that any  $\rho$ -invariant symmetric bilinear form is equivalent to an orthogonal sum of  $\sigma$ -invariant symmetric bilinear forms. We can therefore construct a representation  $\mathbf{Z}/p \rightarrow SO_{2m}(\mathbf{Q})$  without fixed points, what contradicts the first part of the proof.

The rest of the section is devoted to the computation of  $l_{\mathbf{Q}}(p)$ ,  $p$  odd prime.

(2.4) PROPOSITION. 
$$l_{\mathbf{Q}}(p) = \begin{cases} 2 & \text{if } p \not\equiv 7 \pmod{8} \\ 4 & \text{otherwise.} \end{cases}$$

*Proof.* For each  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ , the discriminant of  $\gamma_a$  is not a square in  $\mathbf{Q}$  (cf. lemma (1.2)). Therefore  $l_{\mathbf{Q}}(p)$  must be even. The 4-fold orthogonal sum of a  $\mathbf{Q}$ -bilinear form is equivalent to the standard bilinear form, since every integer is sum of four squares. Let  $C$  be the matrix considered in the proof of lemma (1.2). If it is possible to find two rational numbers  $u$  and  $v$  such that the matrix  $X_{u,v}$

$$X_{u,v} := \begin{bmatrix} uC & 0 \\ 0 & vC \end{bmatrix}$$

represents a bilinear form  $\xi_{u,v}$  which is equivalent to the standard one, then the representation  $\sigma \oplus \sigma$  is equivalent to an orthogonal representation. This sufficient condition is also necessary if  $p \equiv 3 \pmod{4}$  (cf. [5]). For a prime  $p$ , let  $\mathbf{Q}_p$  be the field of  $p$ -adic numbers and write  $\mathbf{Q}_{\infty}$  for  $\mathbf{R}$  as usual. For  $a, b \in \mathbf{Q}$  and for  $v = 2, 3, 5, 7, \dots, \infty$  we write  $(a, b)_v$  for the Hilbert symbol of  $a$  and  $b$  relatively to  $\mathbf{Q}_v$ . For a bilinearform  $\alpha$  given in an orthogonal base by the diagonal matrix

$$\begin{bmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{bmatrix}$$

we write  $H_v(\alpha)$  ( $v = 2, 3, \dots, \infty$ ) for the Hasse invariant, which is defined by

$$H_v(\alpha) = \prod_{i < j} (a_i, a_j)_v$$

Using the formulas given for example by [9] to compute the Hilbert symbol, one check that:

$$\begin{aligned} H_v(\xi_{1,1}) &= 1 & \text{if } p \not\equiv 3 \pmod{4} & & \text{for } v = 2, 3, 5, 7, \dots, \infty, \\ H_2(\xi_{u,v}) &= -1 & \text{if } p \equiv 7 \pmod{8} & & \text{for any } u \text{ and any } v, \\ H_v(\xi_{2p,1}) &= 1 & \text{if } p \equiv 1 \pmod{8} & & \text{for } v = 2, 3, 5, 7, \dots, \infty. \end{aligned}$$

Since the discriminant of  $\xi_{u,v}$  is  $1 \in \mathbf{Q}/\mathbf{Q}^{*2}$  and since  $\xi_{u,v}$  is positive definit for any  $u$  and any  $v$ , it follows that  $\sigma \oplus \sigma$  is equivalent to an orthogonal representation if and only if  $p \not\equiv 7 \pmod{8}$ . It remains to show that, for  $p \equiv 7 \pmod{8}$ , the  $2n$ -fold orthogonal sum  $\mu$  given by the matrix  $H$ :

$$H := \begin{bmatrix} u_1 C & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & u_{2n} C \end{bmatrix}$$

is isomorphic to the standard bilinear form if and only if  $n$  is even. Let  $u_{\text{odd}}$  and  $u_{\text{even}}$  defined by:

$$u_{\text{even}} := \prod_{k=1}^n u_{2k} \quad u_{\text{odd}} := \prod_{k=1}^n u_{2k-1};$$

an easy computation shows that  $H_v(\xi_{u_{\text{even}}, u_{\text{odd}}}) = H_v(\mu)$  if  $n$  is odd. The proposition follows.

### 3. PROOF OF THE MAIN THEOREM

(3.1) LEMMA. Let  $p$  be a prime number ( $p > 2$ ). For every integer  $m$  satisfying  $2m \not\equiv 0 \pmod{(p-1) \cdot l_{\mathbf{Q}}(p)}$  we have  $F_{\mathbf{Q}}(m, p) = 1$ .

*Proof.* Let  $G$  be a  $p$ -group,  $p > 2$ . It follows from (2.2) that any representation  $\rho$  of  $G$  splits:  $\rho = 1 \oplus \tau$  ( $1$  is the 1-dimensional representation of  $G$ ). Then we have  $e(\rho) = e(1)e(\tau) = 0$ .

We are now able to prove the main theorem. It has been showed in [3] that  $F_{\mathbf{Q}}(n) = 4$  if  $n$  is odd. If  $n$  is even, four cases have to be distinguished. If  $p = 2$  then the  $n/2^{N-2}$ -fold sum of the irreducible faithful representation of  $\mathbf{Z}/2^N$ , where  $2^N$  is the 2-primary part of  $\text{den}(B_n/n)$ , is an orthogonal representation with Euler class of order  $2^N$  (cf. [1]). Let now  $p$  be an odd prime. Since the irreducible faithful representation  $v$  of  $\mathbf{Z}/p^r$  ( $r \geq 1$ ) is induced by the irreducible faithful representation of  $\mathbf{Z}/p \subset \mathbf{Z}/p^r$ , the  $M$ -fold sum of  $v$  is equivalent to an orthogonal representation if and only if  $l_{\mathbf{Q}}(p)$  divides  $M$ . Write  $n = Np^k(p-1)$  with  $\text{g.c.d.}(N, p) = 1$ . If  $N$  is even, the  $2N$ -fold sum of the irreducible faithful representation of  $\mathbf{Z}/p^{k+1}$  is orthogonal and has Euler class of order  $p^{k+1}$  (cf. [1]); if  $N$  is odd and  $p \not\equiv 7 \pmod{8}$  then the  $2N$ -fold sum of the irreducible faithful representation of  $\mathbf{Z}/p^{k+1}$  is orthogonal and has Euler class of order  $p^{k+1}$  (cf. [1]). In the three cases, the statement follows from the well known characterization of  $\text{den}(B_n/n)$  (cf. [1] for example). Eventually, applying (3.1) we see that  $F_{\mathbf{Q}}(n, p) = 1$  if  $N$  is odd and  $p \equiv 7 \pmod{8}$ .

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