

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 34 (1988)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS  
**Autor:** Piveteau, Jean-Marc  
**DOI:** <https://doi.org/10.5169/seals-56597>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 18.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS

by Jean-Marc PIVETEAU

If  $\rho: G \rightarrow SO_n(\mathbf{R})$  is an orthogonal representation of the group  $G$ , then the Euler class  $e(\rho)$  is defined as Euler class of the flat real vector bundle over  $BG$  associated with  $\rho$ . For representations of finite groups over a number field  $\mathbf{K}$  there is a uniform bound, depending on  $\mathbf{K}$  and on the degree of the representation only, for the order of the Euler class. This bound has been extensively studied by Eckmann and Mislin ([1], [2], [3]). In this note we discuss analogous bounds for orthogonal representations over the field  $\mathbf{Q}$  of rational numbers. Since the best upper bound for odd dimensional representations is equal to two (cf. [3]), we consider the case of even dimensional  $\mathbf{Q}$ -representations. We will write  $F_{\mathbf{Q}}(m)$  for the best upper bound for the order of the Euler Class  $e(\rho)$ , where  $\rho$  ranges over all  $2m$ -dimensional representations of finite groups over  $\mathbf{Q}$ . Thus, for every representation  $\rho: G \rightarrow SO_{2m}(\mathbf{Q})$  of any finite group  $G$ , it follows that  $F_{\mathbf{Q}}(m) \cdot e(\rho) = 0 \in H^{2m}(G; \mathbf{Z})$ , and  $F_{\mathbf{Q}}(m)$  is the best possible. The prime factorisation of the numbers  $F_{\mathbf{Q}}(m)$  is given as follows:

MAIN THEOREM. *For odd  $m$  we have  $F_{\mathbf{Q}}(m) = 4$ . For even  $m$ , if we write  $F_{\mathbf{Q}}(m, p)$  for the  $p$ -primary part of  $F_{\mathbf{Q}}(m)$  ( $p$ : prime), we have:*

$$F_{\mathbf{Q}}(m, p) = \begin{cases} 1, & \text{if } n \not\equiv 0 \pmod{p-1} \text{ or if } n = Np^k(p-1) \text{ with} \\ & \text{g.c.d. } (p, N) = 1, \text{ } N \text{ odd and } p = 7 \pmod{8}, \\ p\text{-primary part of } \text{den}(B_m/m) & \text{otherwise,} \end{cases}$$

where  $B_m$  is the  $m$ -th Bernoulli-number and  $\text{den}(B_m/m)$  is the denominator of  $B_m/m$  written in its lowest terms.

Note that  $F_{\mathbf{Q}}(m)$  is a lower bound for the order of the universal profinite Euler class  $\hat{e}_{2m}(\mathbf{Q})$  considered by Eckmann and Mislin in [3].

The two first sections contain preliminary results about bilinear forms and orthogonal representations. In the last section, we prove the main theorem.

This paper is a summary of some results of the thesis [8] I have written under the direction of Guido Mislin. I want to express him on this

occasion my gratefulness for his stimulating advices and the interest he constantly showed for this work.

### 1. INVARIANT BILINEAR FORMS

Let  $\mathbf{K}$  be a field of characteristic 0,  $V$  a finite dimensional vector space over  $\mathbf{K}$  and  $\rho: G \rightarrow GL(V)$  a  $\mathbf{K}$ -representation of the group  $G$ . A  $\mathbf{K}$ -bilinearform  $\alpha: V \times V \rightarrow \mathbf{K}$  is called  $\rho$ -invariant if

$$\alpha(\rho(g)x, \rho(g)y) = \alpha(x, y) \quad \forall x, y \in V, \quad \forall g \in G.$$

If  $G$  is finite, then for any bilinear form  $\gamma$  the form  $\bar{\gamma}$  defined by

$$\bar{\gamma}(x, y) := \sum_{g \in G} \gamma(\rho(g)x, \rho(g)y)$$

is  $\rho$ -invariant.

(1.1) *Remark.* If  $\alpha$  is definit (i.e.  $\alpha(x, x) = 0 \Rightarrow x = 0$ ) and if  $\rho$  splits in a direct sum  $\rho = \rho_1 \oplus \rho_2$ , the restriction  $\rho'$  of  $\rho$  to the orthogonal complement of the invariant space corresponding to  $\rho_1$  is equivalent to  $\rho_2$ . Since we always can substitute a representation or a bilinear form by an equivalent one, we can assume that the representation space of a sum is an orthogonal sum of corresponding invariant subspaces.

We call *standard bilinear form* (of dimension  $m$ ) the map  $\beta_m: \mathbf{K}^m \times \mathbf{K}^m \rightarrow \mathbf{K}$  given by

$$\beta_m(x, y) := \sum_{i=1}^m x_i y_i \quad \text{with} \quad x = (x_1, \dots, x_m) \quad \text{and} \quad y = (y_1, \dots, y_m).$$

The group  $O_m(\mathbf{K})$  is the subgroup of  $GL_m(\mathbf{K})$  of matrices  $(a_{ij})$  such that  $\sum_k a_{ik} a_{jk} = \delta_{ij}$  for all  $i, j$ . The group  $SO_m(\mathbf{K})$  is the subgroup of  $O_m(\mathbf{K})$  of matrices  $(a_{ij})$  with  $\det(a_{ij}) = 1$ . It is therefore evident that a representation  $\rho: G \rightarrow GL_m(\mathbf{K})$  is realizable over  $O_m(\mathbf{K})$  if and only if there is a  $\rho$ -invariant symmetric bilinear form which is equivalent to the standard bilinear form.

Let  $p$  be a prime number. Up to equivalence, there is a unique irreducible faithful  $\mathbf{Q}$ -representation  $\sigma$  of  $\mathbf{Z}/p$ ; it is given by

$$\begin{aligned} \sigma: \mathbf{Z}/p &\rightarrow GL_{p-1}(\mathbf{Q}) \\ 1 &\mapsto A := \begin{bmatrix} 0 & \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & -1 \\ & & \cdot & & \\ \cdot & \cdot & \cdot & 1 & -1 \end{bmatrix} \end{aligned}$$

We can identify the irreducible faithful  $\mathbf{Q}[\mathbf{Z}/p]$ -Module  $\mathbf{Q}^{p-1}$  with  $\mathbf{Q}(\zeta_p)$  ( $\zeta_p$ : primitive  $p$ -th root of unity,  $1 \in \mathbf{Z}/p$  acts on  $\mathbf{Q}(\zeta_p)$  by multiplication with  $\zeta_p$ ). Any symmetric  $\sigma$ -invariant bilinear form is given by  $tr_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(axy\bar{y})$  with  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$  (cf. [4] or [6]). We write  $\gamma_a$  for the  $\sigma$ -invariant bilinear form corresponding to  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ .

(1.2) LEMMA. *The discriminant of  $\gamma_a$  in  $\mathbf{Q}/\mathbf{Q}^{*2}$  is equal to  $p \bmod \mathbf{Q}^{*2}$ .*

*Proof.* Since  $a \in \mathbf{L} := \mathbf{Q}(\zeta_p + \zeta_p^{-1})$  we have:  $\gamma_a = tr_{\mathbf{L}/\mathbf{Q}}(tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}axy\bar{y})$ . An easy computation shows that  $tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}(axy\bar{y})$  is a 2-dimensional symmetric  $\mathbf{L}$ -bilinearform with discriminant  $4 - (\zeta_p + \zeta_p^{-1})^2 \bmod \mathbf{L}^{*2} \in \mathbf{L}/\mathbf{L}^{*2}$ . Applying [7, Lemma 2.2] we conclude that the discriminant of  $\gamma_a$  is independant of  $a \in \mathbf{L}$ . Consider now the matrix representation of  $\sigma$  given before ( $\sigma$ : irreducible faithful  $\mathbf{Q}$ -representation of  $\mathbf{Z}/p$ ). Let  $C$  be the  $(p-1) \times (p-1)$ -matrix given by:

$$C := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \cdot & \cdot \\ & & & -1 & 2 \end{bmatrix}$$

It is easy to check that  $C$  is the matrix of a  $\sigma$ -invariant symmetric bilinear form. The Lemma follows since the determinant of  $C$  is equal to  $p$ .

## 2. ORTHOGONAL REPRESENTATIONS OF $p$ -GROUPS

Let  $p > 2$  be an odd prime. The integer  $l_{\mathbf{Q}}(p)$  is defined by

$$l_{\mathbf{Q}}(p) := \text{g.c.d.} \left\{ \begin{array}{l} m > 1 \\ \left. \begin{array}{l} \text{the } m\text{-fold direct sum } \sigma \oplus \dots \oplus \sigma \text{ of the irre-} \\ \text{ducible faithful } \mathbf{Q}\text{-representation } \sigma \text{ of } \mathbf{Z}/p \text{ is} \\ \text{equivalent to an orthogonal representation} \end{array} \right\} \end{array} \right.$$

The importance played by cyclic groups in the investigation of representations of  $p$ -groups is given by the following result (cf. [1, Theorem (1.10)]):

(2.1) PROPOSITION. *Let  $G$  be a finite  $p$ -group ( $p > 2$ ) and let  $\rho$  be an irreducible  $\mathbf{Q}$ -representation of  $G$ . Then either  $\rho$  is induced from a representation  $\theta$  of a normal subgroup of index  $p$ , or  $\rho$  factors through a  $\mathbf{Q}$ -representation of  $\mathbf{Z}/p$ .*

The degree of an irreducible non trivial  $\mathbf{Q}$ -representation of a finite  $p$ -group is therefore of the form  $p^k(p-1)$  ( $k=0, 1, 2, \dots$ ), cf. [1, Corollary (1.11)].

(2.2) PROPOSITION. *Let  $G$  be a  $p$ -group ( $p > 2$ ) and  $\rho: G \rightarrow SO_{2m}(\mathbf{Q})$  a representation of  $G$  with  $2m \not\equiv 0 \pmod{l_{\mathbf{Q}}(p) \cdot (p-1)}$ . Then  $\rho$  has a fixed point (i.e.  $\rho = 1 \oplus \tau$  where  $1$  is the unique 1-dimensional  $\mathbf{Q}$ -representation of  $G$ ).*

We will need the following lemma for the proof of (2.2):

(2.3) LEMMA. *Let  $\rho: G \rightarrow GL_m(\mathbf{Q})$  be an irreducible non trivial representation of the  $p$ -group  $G$  ( $p > 2$ ) and let  $\psi$  be a  $\rho$ -invariant symmetric bilinear form. If we write  $\sigma$  for the irreducible faithful representation of  $\mathbf{Z}/p$ , then there exist  $\sigma$ -invariant bilinear forms  $\Gamma_1, \dots, \Gamma_s$  such that  $\psi$  is equivalent to the orthogonal sum  $\Gamma_1 \perp \dots \perp \Gamma_s$ .*

*Proof.* Let  $p^k(p-1)$  be the degree of  $\rho$ . We prove the lemma by induction on  $k$ . For  $k = 0$ ,  $\rho$  factors through the irreducible faithful representation  $\sigma$  of  $\mathbf{Z}/p$ . Every  $\rho$ -invariant symmetric bilinear form  $\psi$  is therefore  $\sigma$ -invariant. For  $k > 0$ ,  $\rho$  is induced by a representation  $\theta$  of a normal subgroup  $H$  of index  $p$ . The restriction  $\rho_H$  of  $\rho$  to  $H$  splits in a direct sum:  $\rho = \theta_1 \oplus \dots \oplus \theta_p$  with  $\theta = \theta_1$  and  $\theta_i$  is irreducible for  $i = 1, \dots, p$ . By (1.1) we can assume that  $\mathbf{Q}^m$  is the orthogonal sum of the corresponding irreducible invariant subspaces. The assertion follows by induction.

*Proof of (2.2).* If  $G = \mathbf{Z}/p$ , we split  $\rho$  in a direct sum:  $\rho = n_0 1 \oplus n_1 \sigma$  ( $1$ : one dimensional representation of  $\mathbf{Z}/p$ ;  $\sigma$ : irreducible faithful representation of  $\mathbf{Z}/p$ ). If  $n_0 = 0$  then  $n_1$  must be a multiple of  $l_{\mathbf{Q}}(p)$ , i.e. we have  $2m \equiv 0 \pmod{(p-1)l_{\mathbf{Q}}(p)}$ . Contradiction.

If  $G$  is not  $\mathbf{Z}/p$ , we split  $\rho$  in a direct sum of irreducible representations:  $\rho = \rho_1 \oplus \dots \oplus \rho_t$ , chosen in such a way that  $\mathbf{Q}^{2m}$  is the orthogonal sum of the corresponding invariant subspaces. Suppose now that  $\rho$  has no fixed points. Then all  $\rho_i$  are non trivial and it follows from (2.3) that any  $\rho$ -invariant symmetric bilinear form is equivalent to an orthogonal sum of  $\sigma$ -invariant symmetric bilinear forms. We can therefore construct a representation  $\mathbf{Z}/p \rightarrow SO_{2m}(\mathbf{Q})$  without fixed points, what contradicts the first part of the proof.

The rest of the section is devoted to the computation of  $l_{\mathbf{Q}}(p)$ ,  $p$  odd prime.

(2.4) PROPOSITION.  $l_{\mathbf{Q}}(p) = \begin{cases} 2 & \text{if } p \not\equiv 7 \pmod{8} \\ 4 & \text{otherwise.} \end{cases}$

*Proof.* For each  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ , the discriminant of  $\gamma_a$  is not a square in  $\mathbf{Q}$  (cf. lemma (1.2)). Therefore  $l_{\mathbf{Q}}(p)$  must be even. The 4-fold orthogonal sum of a  $\mathbf{Q}$ -bilinear form is equivalent to the standard bilinear form, since every integer is sum of four squares. Let  $C$  be the matrix considered in the proof of lemma (1.2). If it is possible to find two rational numbers  $u$  and  $v$  such that the matrix  $X_{u,v}$

$$X_{u,v} := \begin{bmatrix} uC & 0 \\ 0 & vC \end{bmatrix}$$

represents a bilinear form  $\xi_{u,v}$  which is equivalent to the standard one, then the representation  $\sigma \oplus \sigma$  is equivalent to an orthogonal representation. This sufficient condition is also necessary if  $p = 3 \pmod 4$  (cf. [5]). For a prime  $p$ , let  $\mathbf{Q}_p$  be the field of  $p$ -adic numbers and write  $\mathbf{Q}_\infty$  for  $\mathbf{R}$  as usual. For  $a, b \in \mathbf{Q}$  and for  $v = 2, 3, 5, 7, \dots, \infty$  we write  $(a, b)_v$  for the Hilbert symbol of  $a$  and  $b$  relatively to  $\mathbf{Q}_v$ . For a bilinearform  $\alpha$  given in an orthogonal base by the diagonal matrix

$$\begin{bmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{bmatrix}$$

we write  $H_v(\alpha)$  ( $v = 2, 3, \dots, \infty$ ) for the Hasse invariant, which is defined by

$$H_v(\alpha) = \prod_{i < j} (a_i, a_j)_v$$

Using the formulas given for example by [9] to compute the Hilbert symbol, one check that:

$$\begin{aligned} H_v(\xi_{1,1}) &= 1 && \text{if } p \neq 3 \pmod 4 && \text{for } v = 2, 3, 5, 7, \dots, \infty, \\ H_2(\xi_{u,v}) &= -1 && \text{if } p = 7 \pmod 8 && \text{for any } u \text{ and any } v, \\ H_v(\xi_{2p,1}) &= 1 && \text{if } p = 1 \pmod 8 && \text{for } v = 2, 3, 5, 7, \dots, \infty. \end{aligned}$$

Since the discriminant of  $\xi_{u,v}$  is  $1 \in \mathbf{Q}/\mathbf{Q}^{*2}$  and since  $\xi_{u,v}$  is positive definit for any  $u$  and any  $v$ , it follows that  $\sigma \oplus \sigma$  is equivalent to an orthogonal representation if and only if  $p \neq 7 \pmod 8$ . It remains to show that, for  $p = 7 \pmod 8$ , the  $2n$ -fold orthogonal sum  $\mu$  given by the matrix  $H$ :

$$H := \begin{bmatrix} u_1 C & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & u_{2n} C \end{bmatrix}$$

is isomorphic to the standard bilinear form if and only if  $n$  is even. Let  $u_{\text{odd}}$  and  $u_{\text{even}}$  defined by:

$$u_{\text{even}} := \prod_{k=1}^n u_{2k} \quad u_{\text{odd}} := \prod_{k=1}^n u_{2k-1};$$

an easy computation shows that  $H_v(\xi_{u_{\text{even}}, u_{\text{odd}}}) = H_v(\mu)$  if  $n$  is odd. The proposition follows.

### 3. PROOF OF THE MAIN THEOREM

(3.1) LEMMA. Let  $p$  be a prime number ( $p > 2$ ). For every integer  $m$  satisfying  $2m \not\equiv 0 \pmod{(p-1) \cdot l_{\mathbf{Q}}(p)}$  we have  $F_{\mathbf{Q}}(m, p) = 1$ .

*Proof.* Let  $G$  be a  $p$ -group,  $p > 2$ . It follows from (2.2) that any representation  $\rho$  of  $G$  splits:  $\rho = 1 \oplus \tau$  ( $1$  is the 1-dimensional representation of  $G$ ). Then we have  $e(\rho) = e(1)e(\tau) = 0$ .

We are now able to prove the main theorem. It has been showed in [3] that  $F_{\mathbf{Q}}(n) = 4$  if  $n$  is odd. If  $n$  is even, four cases have to be distinguished. If  $p = 2$  then the  $n/2^{N-2}$ -fold sum of the irreducible faithful representation of  $\mathbf{Z}/2^N$ , where  $2^N$  is the 2-primary part of  $\text{den}(B_n/n)$ , is an orthogonal representation with Euler class of order  $2^N$  (cf. [1]). Let now  $p$  be an odd prime. Since the irreducible faithful representation  $\nu$  of  $\mathbf{Z}/p^r$  ( $r \geq 1$ ) is induced by the irreducible faithful representation of  $\mathbf{Z}/p \subset \mathbf{Z}/p^r$ , the  $M$ -fold sum of  $\nu$  is equivalent to an orthogonal representation if and only if  $l_{\mathbf{Q}}(p)$  divides  $M$ . Write  $n = Np^k(p-1)$  with  $\text{g.c.d.}(N, p) = 1$ . If  $N$  is even, the  $2N$ -fold sum of the irreducible faithful representation of  $\mathbf{Z}/p^{k+1}$  is orthogonal and has Euler class of order  $p^{k+1}$  (cf. [1]); if  $N$  is odd and  $p \not\equiv 7 \pmod{8}$  then the  $2N$ -fold sum of the irreducible faithful representation of  $\mathbf{Z}/p^{k+1}$  is orthogonal and has Euler class of order  $p^{k+1}$  (cf. [1]). In the three cases, the statement follows from the well known characterization of  $\text{den}(B_n/n)$  (cf. [1] for example). Eventually, applying (3.1) we see that  $F_{\mathbf{Q}}(n, p) = 1$  if  $N$  is odd and  $p \equiv 7 \pmod{8}$ .

## REFERENCES

- [1] ECKMANN, B. and G. MISLIN. Rational representations of finite groups and their Euler class. *Math. Ann.* 245 (1979), 45-54.
- [2] ECKMANN, B. and G. MISLIN. On the Euler class of representations of finite groups over real fields. *Comment. Math. Helvetici* 55 (1980), 319-329.
- [3] ECKMANN, B. and G. MISLIN. Galois action on algebraic matrix groups, Chern classes and the Euler class. *Math. Ann.* 271 (1985), 349-358.
- [4] FEIT, W. On integral representations of finite groups. *Proc. London Math. Soc.* 29 (1974), 633-683.
- [5] ——— On certain rational quadratic forms. *Linear and Multilinear Algebra* 3 (1975), 25-32.
- [6] FRÖHLICH, A. and A. M. MCEVETT. The representation of groups by automorphisms of forms. *J. Algebra* 12 (1969), 114-133.
- [7] MILNOR, J. On isometries of inner product spaces. *Invent. Math.* 8 (1969), 83-97.
- [8] PIVETEAU, J.-M. *Über die Eulerklasse orthogonaler Darstellungen endlicher Gruppen.* Diss. ETH Nr. 8373 (1987).
- [9] SERRE, J.-P. *Cours d'arithmétique.* Presses Universitaires de France. 1977.

(Reçu le 4 décembre 1987)

Jean-Marc Piveteau

Department of Mathematics  
ETH — Zürich  
CH - 8092 Zürich (Switzerland)

**vide-leer-empty**