**Zeitschrift:** L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 34 (1988)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE EULER CLASS OF ORTHOGONAL RATIONAL

REPRESENTATIONS OF FINITE GROUPS

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**DOI:** https://doi.org/10.5169/seals-56597

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# THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS

# by Jean-Marc PIVETEAU

If  $\rho: G \to SO_n(\mathbf{R})$  is an orthogonal representation of the group G, then the Euler class  $e(\rho)$  is defined as Euler class of the flat real vector bundle over BG associated with  $\rho$ . For representations of finite groups over a number field  $\mathbf{K}$  there is a uniform bound, depending on  $\mathbf{K}$  and on the degree of the representation only, for the order of the Euler class. This bound has been extensively studied by Eckmann and Mislin ([1], [2], [3]). In this note we discuss analogous bounds for orthogonal representations over the field  $\mathbf{Q}$  of rational numbers. Since the best upper bound for odd dimensional representations is equal to two (cf. [3]), we consider the case of even dimensional  $\mathbf{Q}$ -representations. We will write  $F_{\mathbf{Q}}(m)$  for the best upper bound for the order of the Euler Class  $e(\rho)$ , where  $\rho$  ranges over all 2m-dimensional representations of finite groups over  $\mathbf{Q}$ . Thus, for every representation  $\rho: G \to SO_{2m}(\mathbf{Q})$  of any finite group G, it follows that  $F_{\mathbf{Q}}(m) \cdot e(\rho) = 0 \in H^{2m}(G; \mathbf{Z})$ , and  $F_{\mathbf{Q}}(m)$  is the best possible. The prime factorisation of the numbers  $F_{\mathbf{Q}}(m)$  is given as follows:

Main Theorem. For odd m we have  $F_{\mathbf{Q}}(m)=4$ . For even m, if we write  $F_{\mathbf{Q}}(m,p)$  for the p-primary part of  $F_{\mathbf{Q}}(m)$  (p: prime), we have:

$$F_{\mathbf{Q}}(m,p) = \left\{ \begin{array}{ll} 1, & \mbox{if} & \mbox{$n \neq 0$ mod } (p-1)$ or if $n = Np^k(p-1)$ with \\ & \mbox{g.c.d. } (p,N) = 1, & N \mbox{ odd and } p = 7 \mbox{ mod } 8, \\ & \mbox{$p$-primary part of } \mbox{den} \left(B_m/m\right) \mbox{ otherwise,} \end{array} \right.$$

where  $B_m$  is the m-th Bernoulli-number and  $den(B_m/m)$  is the denominator of  $B_m/m$  written in its lowest terms.

Note that  $F_{\mathbf{Q}}(m)$  is a lower bound for the order of the universal profinite Euler class  $\hat{e}_{2m}(\mathbf{Q})$  considered by Eckmann and Mislin in [3].

The two first sections contain preliminary results about bilinear forms and orthogonal representations. In the last section, we prove the main theorem.

This paper is a summary of some results of the thesis [8] I have written under the direction of Guido Mislin. I want to express him on this

occasion my gratefulness for his stimulating advices and the interest he constantly showed for this work.

# 1. INVARIANT BILINEAR FORMS

Let **K** be a field of characteristic 0, V a finite dimensional vector space over **K** and  $\rho: G \to GL(V)$  a **K**-representation of the group G. A **K**-bilinearform  $\alpha: V \times V \to \mathbf{K}$  is called  $\rho$ -invariant if

$$\alpha(\rho(g)x, \rho(g)y) = \alpha(x, y) \quad \forall x, y \in V, \quad \forall g \in G.$$

If G is finite, then for any bilinear form  $\gamma$  the form  $\bar{\gamma}$  defined by

$$\bar{\gamma}(x, y) := \sum_{g \in G} \gamma(\rho(g)x, \rho(g)y)$$

is ρ-invariant.

(1.1) Remark. If  $\alpha$  is definit (i.e.  $\alpha(x, x) = 0 \Rightarrow x = 0$ ) and if  $\rho$  splits in a direct sum  $\rho = \rho_1 \oplus \rho_2$ , the restriction  $\rho'$  of  $\rho$  to the orthogonal complement of the invariant space corresponding to  $\rho_1$  is equivalent to  $\rho_2$ . Since we always can substitute a representation or a bilinear form by an equivalent one, we can assume that the representation space of a sum is an orthogonal sum of corresponding invariant subspaces.

We call standard bilinear form (of dimension m) the map  $\beta_m \colon \mathbf{K}^m \times \mathbf{K}^m \to \mathbf{K}$  given by

$$\beta_m(x, y) := \sum_{i=1}^m x_i y_i$$
 with  $x = (x_1, ..., x_m)$  and  $y = (y_1, ..., y_m)$ .

The group  $O_m(\mathbf{K})$  is the subgroup of  $GL_m(\mathbf{K})$  of matrices  $(a_{ij})$  such that  $\sum_k a_{ik}a_{jk} = \delta_{ij}$  for all i, j. The group  $SO_m(\mathbf{K})$  is the subgroup of  $O_m(\mathbf{K})$  of matrices  $(a_{ij})$  with  $\det(a_{ij}) = 1$ . It is therefore evident that a representation  $\rho: G \to GL_m(\mathbf{K})$  is realizable over  $O_m(\mathbf{K})$  if and only if there is a  $\rho$ -invariant symmetric bilinear form which is equivalent to the standard bilinear form.

Let p be a prime number. Up to equivalence, there is a unique irreducible faithful **Q**-representation  $\sigma$  of  $\mathbb{Z}/p$ ; it is given by

$$\sigma: \mathbf{Z}/p \to GL_{p-1}(\mathbf{Q})$$

$$1 \mapsto A:= \begin{bmatrix} 0 & . & . & -1 \\ 1 & . & . & -1 \\ & & . & \\ . & . & . & 1 & -1 \end{bmatrix}$$

We can identify the irreducible faithful  $\mathbf{Q}[\mathbf{Z}/p]$ -Module  $\mathbf{Q}^{p-1}$  with  $\mathbf{Q}(\zeta_p)$  ( $\zeta_p$ : primitive p-th root of unity,  $1 \in \mathbf{Z}/p$  acts on  $\mathbf{Q}(\zeta_p)$  by multiplication with  $\zeta_p$ ). Any symmetric  $\sigma$ -invariant bilinear form is given by  $tr_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(ax\bar{y})$  with  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$  (cf. [4] or [6]). We write  $\gamma_a$  for the  $\sigma$ -invariant bilinear form corresponding to  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ .

(1.2) Lemma. The discriminant of  $\gamma_a$  in  $\mathbf{Q}/\mathbf{Q}^{*2}$  is equal to  $p \mod \mathbf{Q}^{*2}$ .

Proof. Since  $a \in \mathbf{L} := \mathbf{Q}(\zeta_p + \zeta_p^{-1})$  we have:  $\gamma_a = tr_{\mathbf{L}/\mathbf{Q}}(tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}ax\bar{y})$ . An easy computation shows that  $tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}(ax\bar{y})$  is a 2-dimensional symmetric **L**-bilinearform with discriminant  $4 - (\zeta_p + \zeta_p^{-1})^2 \mod \mathbf{L}^{*2} \in \mathbf{L}/\mathbf{L}^{*2}$ . Applying [7, Lemma 2.2] we conclude that the discriminant of  $\gamma_a$  is independent of  $a \in \mathbf{L}$ . Consider now the matrix representation of  $\sigma$  given before ( $\sigma$ : irreducible faithful **Q**-representation of  $\mathbf{Z}/p$ ). Let C be the  $(p-1) \times (p-1)$ -matrix given by:

$$C := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 2 \end{bmatrix}$$

It is easy to check that C is the matrix of a  $\sigma$ -invariant symmetric bilinear form. The Lemma follows since the determinant of C is equal to p.

# 2. Orthogonal representations of p-groups

Let p > 2 be an odd prime. The integer  $l_{\mathbf{Q}}(p)$  is defined by

$$l_{\mathbf{Q}}(p)$$
: = g.c.d.  $\left\{\begin{array}{c} m > 1 \\ \text{equivalent to an orthogonal representation} \end{array}\right.$ 

The importance played by cyclic groups in the investigation of representations of p-groups is given by the following result (cf. [1, Theorem (1.10)]):

(2.1) Proposition. Let G be a finite p-group (p>2) and let  $\rho$  be an irreducible  $\mathbf{Q}$ -representation of G. Then either  $\rho$  is induced from a representation  $\theta$  of a normal subgroup of index p, or  $\rho$  factors through a  $\mathbf{Q}$ -representation of  $\mathbf{Z}/p$ .

The degree of an irreducible non trivial **Q**-representation of a finite p-group is therefore of the form  $p^k(p-1)$  (k=0, 1, 2, ...), cf. [1, Corollary (1.11)].

(2.2) PROPOSITION. Let G be a p-group (p>2) and  $\rho: G \to SO_{2m}(\mathbf{Q})$  a representation of G with  $2m \neq 0 \mod (l_{\mathbf{Q}}(p) \cdot (p-1))$ . Then  $\rho$  has a fixed point (i.e.  $\rho = 1 \oplus \tau$  where 1 is the unique 1-dimensional  $\mathbf{Q}$ -representation of G).

We will need the following lemma for the proof of (2.2):

(2.3) Lemma. Let  $\rho: G \to GL_m(\mathbf{Q})$  be an irreducible non trivial representation of the p-group G(p>2) and let  $\psi$  be a  $\rho$ -invariant symmetric bilinear form. If we write  $\sigma$  for the irreducible faithful representation of  $\mathbf{Z}/p$ , then there exist  $\sigma$ -invariant bilinear forms  $\Gamma_1,...,\Gamma_s$  such that  $\psi$  is equivalent to the orthogonal sum  $\Gamma_1 \perp ... \perp \Gamma_s$ .

Proof. Let  $p^k(p-1)$  be the degree of  $\rho$ . We prove the lemma by induction on k. For k=0,  $\rho$  factors through the irreducible faithful representation  $\sigma$  of  $\mathbb{Z}/p$ . Every  $\rho$ -invariant symmetric bilinearform  $\psi$  is therefore  $\sigma$ -invariant. For k>0,  $\rho$  is induced by a representation  $\theta$  of a normal subgroup H of index p. The restriction  $\rho_H$  of  $\rho$  to H splits in a direct sum:  $\rho=\theta_1\oplus\ldots\oplus\theta_p$  with  $\theta=\theta_1$  and  $\theta_i$  is irreducible for  $i=1,\ldots,p$ . By (1.1) we can assume that  $\mathbb{Q}^m$  is the orthogonal sum of the corresponding irreducible invariant subspaces. The assertion follows by induction.

Proof of (2.2). If  $G = \mathbb{Z}/p$ , we split  $\rho$  in a direct sum:  $\rho = n_0 1 \oplus n_1 \sigma$  (1: one dimensional representation of  $\mathbb{Z}/p$ ;  $\sigma$ : irreducible faithful representation of  $\mathbb{Z}/p$ ). If  $n_0 = 0$  then  $n_1$  must be a multiple of  $l_{\mathbb{Q}}(p)$ , i.e. we have  $2m = 0 \mod (p-1)l_{\mathbb{Q}}(p)$ . Contradiction.

If G is not  $\mathbb{Z}/p$ , we split  $\rho$  in a direct sum of irreducible representations:  $\rho = \rho_1 \oplus ... \oplus \rho_t$ , chosen in such a way that  $\mathbb{Q}^{2m}$  is the orthogonal sum of the corresponding invariant subspaces. Suppose now that  $\rho$  has no fixed points. Then all  $\rho_i$  are non trivial and it follows from (2.3) that any  $\rho$ -invariant symmetric bilinear form is equivalent to an orthogonal sum of  $\sigma$ -invariant symmetric bilinear forms. We can therefore construct a representation  $\mathbb{Z}/p \to SO_{2m}(\mathbb{Q})$  without fixed points, what contradicts the first part of the proof.

The rest of the section is devoted to the computation of  $l_{\mathbf{Q}}(p)$ , p odd prime.

(2.4) Proposition. 
$$l_{\mathbf{Q}}(p) = \begin{cases} 2 & \text{if } p \neq 7 \mod 8 \\ 4 & \text{otherwise.} \end{cases}$$

*Proof.* For each  $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ , the discriminant of  $\gamma_a$  is not a square in  $\mathbf{Q}$  (cf. lemma (1.2)). Therefore  $l_{\mathbf{Q}}(p)$  must be even. The 4-fold orthogonal sum of a  $\mathbf{Q}$ -bilinear form is equivalent to the standard bilinear form, since every integer is sum of four squares. Let C be the matrix considered in the proof of lemma (1.2). If it is possible to find two rational numbers u and v such that the matrix  $X_{u,v}$ 

$$X_{u,v} := \begin{bmatrix} uC & 0 \\ 0 & vC \end{bmatrix}$$

represents a bilinear form  $\xi_{u,v}$  which is equivalent to the standard one, then the representation  $\sigma \oplus \sigma$  is equivalent to an orthogonal representation. This sufficient condition is also necessary if  $p = 3 \mod 4$  (cf. [5]). For a prime p, let  $\mathbf{Q}_p$  be the field of p-adic numbers and write  $\mathbf{Q}_{\infty}$  for  $\mathbf{R}$  as usual. For  $a, b \in \mathbf{Q}$  and for  $v = 2, 3, 5, 7, ..., \infty$  we write  $(a, b)_v$  for the Hilbert symbol of a and b relatively to  $\mathbf{Q}_v$ . For a bilinearform  $\alpha$  given in an orthogonal base by the diagonal matrix

$$\begin{bmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & a_n \end{bmatrix}$$

we write  $H_v(\alpha)$   $(v=2, 3, ..., \infty)$  for the Hasse invariant, which is defined by

$$H_v(\alpha) = \prod_{i < j} (a_i, a_j)_v$$

Using the formulas given for example by [9] to compute the Hilbert symbol, one check that:

$$H_v(\xi_{1,\,1}) = 1$$
 if  $p \neq 3 \mod 4$  for  $v = 2, 3, 5, 7, ..., \infty$ ,  $H_2(\xi_{u,\,v}) = -1$  if  $p = 7 \mod 8$  for any  $u$  and any  $v$ ,  $H_v(\xi_{2p,\,1}) = 1$  if  $p = 1 \mod 8$  for  $v = 2, 3, 5, 7, ... \infty$ .

Since the discriminant of  $\xi_{u,v}$  is  $1 \in \mathbb{Q}/\mathbb{Q}^{*2}$  and since  $\xi_{u,v}$  is positive definit for any u and any v, it follows that  $\sigma \oplus \sigma$  is equivalent to an orthogonal representation if and only if  $p \neq 7 \mod 8$ . It remains to show that, for  $p = 7 \mod 8$ , the 2n-fold orthogonal sum  $\mu$  given by the matrix H:

is isomorphic to the standard bilinear form if and only if n is even. Let  $u_{\text{odd}}$  and  $u_{\text{even}}$  defined by:

$$u_{\text{even}} := \prod_{k=1}^{n} u_{2k}$$
  $u_{\text{odd}} := \prod_{k=1}^{n} u_{2k-1};$ 

an easy computation shows that  $H_v(\xi_{u_{\text{even}}, u_{\text{odd}}}) = H_v(\mu)$  if n is odd. The proposition follows.

# 3. Proof of the main theorem

(3.1) Lemma. Let p be a prime number (p>2). For every integer m satisfying  $2m \neq 0 \mod (p-1) \cdot l_{\mathbf{Q}}(p)$  we have  $F_{\mathbf{Q}}(m,p) = 1$ .

*Proof.* Let G be a p-group, p > 2. It follows from (2.2) that any representation  $\rho$  of G splits:  $\rho = 1 \oplus \tau$  (1 is the 1-dimensional representation of G). Then we have  $e(\rho) = e(1)e(\tau) = 0$ .

We are now able to prove the main theorem. It has been showed in [3] that  $F_{\mathbf{Q}}(n) = 4$  if n is odd. If n is even, four cases have to be distinguished. If p = 2 then the  $n/2^{N-2}$ -fold sum of the irreducible faithful representation of  $\mathbb{Z}/2^N$ , where  $2^N$  is the 2-primary part of den $(B_n/n)$ , is an orthogonal representation with Euler class of order  $2^N$  (cf. [1]). Let now p be an odd prime. Since the irreducible faithful representation v of  $\mathbb{Z}/p^r(r \ge 1)$  is induced by the irreducible faithful representation of  $\mathbb{Z}/p \subset \mathbb{Z}/p^r$ , the M-fold sum of v is equivalent to an orthogonal representation if and only if  $l_0(p)$  divides M. Write  $n = Np^k(p-1)$  with g.c.d. (N, p) = 1. If N is even, the 2N-fold sum of the irreducible faithful representation of  $\mathbb{Z}/p^{k+1}$  is orthogonal and has Euler class of order  $p^{k+1}$  (cf. [1]); if N is odd and  $p \neq 7 \mod 8$  then the 2N-fold sum of the irreducible faithful representation of  $\mathbb{Z}/p^{k+1}$  is orthogonal and has Euler class of order  $p^{k+1}$  (cf. [1]). In the three cases, the statement follows from the well known characterization of  $den(B_n/n)$  (cf. [1] for example). Eventually, applying (3.1) we see that  $F_{\mathbf{Q}}(n, p) = 1$  if N is odd and  $p = 7 \mod 8$ .

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(Reçu le 4 décembre 1987)

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