

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 34 (1988)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** GEODESICS IN THE UNIT TANGENT BUNDLE OF A ROUND SPHERE  
**Autor:** Gluck, Herman  
**Kapitel:** 4. SASAKI'S EQUATIONS  
**DOI:** <https://doi.org/10.5169/seals-56596>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 28.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

The formulas for curvature, torsion and writhe are as follows.

$$\text{Curvature} = \kappa = \sqrt{(a^2 - 1)(1 - b^2)}$$

$$\text{Torsion} = \tau = ab$$

$$\text{Writhe} = \rho = \sqrt{a^2 + b^2 - 1}.$$

Consider the 3-dimensional linear space of vector fields

$$aT(t) + bN(t) + cB(t)$$

which can be written as constant coefficient combinations of the Frenet vectors along the helix  $p(t)$ . Covariant differentiation along the helix maps this linear space to itself according to the Frenet formulas.

We've already noted in the introduction that the instantaneous axis vector  $U = \tau T - \kappa B$  satisfies  $U' = 0$ .

Consider the vectors  $N$  and  $V = (\kappa/\rho)T + (\tau/\rho)B$ , which form an orthonormal basis for the orthogonal complement of  $U$ . Note that

$$N' = -\kappa T - \tau B = -\rho V, \quad \text{and}$$

$$V' = (\kappa/\rho)T' + (\tau/\rho)B' = (\kappa/\rho)(\kappa N) + (\tau/\rho)(\tau N) = \rho N.$$

Thus, covariant differentiation along the helix kills the instantaneous axis vector and takes the orthogonal 2-plane to itself by a  $90^\circ$  rotation, followed by multiplication by the writhe.

#### 4. SASAKI'S EQUATIONS

Let  $M$  be any Riemannian manifold, and  $UM$  its unit tangent bundle with the Riemannian metric described in section 1.

**THEOREM** (Sasaki [Sa], 1958). *The curve  $(p(t), v(t))$  in  $UM$  is a constant speed geodesic there if and only if both of the following equations hold:*

$$1) \quad v'' = -\langle v', v' \rangle v$$

$$2) \quad p'' = R(v', v)p'.$$

Here, primes denote ordinary derivatives with respect to  $t$  when applied to functions, and covariant derivatives along the path  $p(t)$  when applied to vector fields. For example, the first prime in  $p''$  represents ordinary differentiation, the second, covariant differentiation. The symbol  $R$  denotes the Riemann curvature transformation

$$R: TM_p \times TM_p \rightarrow \text{Hom}(TM_p, TM_p).$$

We give a quick proof of Sasaki's theorem, and refer the reader interested in further details both to Sasaki's original paper and to a brief treatment of his result in [Ba-Br-Bu, pages 37-39].

First note that the energy of the curve  $(p(t), v(t))$  in  $UM$  is given by

$$E = 1/2 \int_0^1 \langle p', p' \rangle dt + 1/2 \int_0^1 \langle v', v' \rangle dt.$$

This curve is a geodesic in  $UM$  precisely when it is a critical point of  $E$  for fixed end point variations. These include variations which fix all the foot points  $p(t)$ , that is, fixed end point variations of the second integral. This second integral equals the energy of the curve  $u(t)$ , lying in the unit sphere in the tangent space to  $M$  at  $p(0)$ , obtained by parallel translating  $v(t)$  backwards along  $p(t)$  to  $p(0)$ . Hence the curve  $u(t)$  is a geodesic, that is, a great circle arc, in this unit sphere.

Because  $u(t)$  is a unit vector field,  $\langle u, u \rangle = 1$ . Differentiating twice,  $\langle u'', u \rangle + \langle u', u' \rangle = 0$ . Because  $u(t)$  runs at constant speed along a great circle,  $u''$  is parallel to  $u$ . Hence  $u'' = -\langle u', u' \rangle u$ . Parallel translating this equation back out along  $p(t)$ , we get Sasaki's first equation.

To get Sasaki's second equation, consider a fixed end point variation  $(p(t, s), v(t, s))$  of the curve  $(p(t), v(t))$  in  $UM$ . Suppose this curve is a critical point of the energy  $E$  for such variations. Then

$$0 = dE/ds = 1/2 \int_0^1 \partial/\partial s \langle p', p' \rangle dt + 1/2 \int_0^1 \partial/\partial s \langle v', v' \rangle dt.$$

The first integrand is processed by differentiating with respect to  $s$ , then interchanging the order of the  $t$  and  $s$  differentiations, and finally setting up for integration by parts, yielding

$$\partial/\partial t \langle \partial p/\partial s, p' \rangle - \langle \partial p/\partial s, p'' \rangle.$$

The second integrand is processed similarly, except that the Riemann curvature transformation appears as a penalty for interchanging the order of the  $t$  and  $s$  differentiations, since this time both are covariant. We get

$$\partial/\partial t \langle \partial v/\partial s, v' \rangle - \langle \partial v/\partial s, v'' \rangle + \langle R(\partial p/\partial s, p')v, v' \rangle.$$

Integrating these two expressions with respect to  $t$ , as required, the leading term of each drops out because the variation is fixed end point. Furthermore, the second term of the second expression is dead zero: since  $\langle v, v \rangle = 1$ ,

$\partial v/\partial s$  is orthogonal to  $v$ , while by Sasaki's first equation,  $v''$  is parallel to  $v$ . We get

$$0 = \int_0^1 \langle \partial p/\partial s, p'' \rangle - \langle R(\partial p/\partial s, p')v, v' \rangle dt.$$

Capitalizing on the symmetries of the curvature, we rewrite this as

$$0 = \int_0^1 \langle p'' - R(v', v)p', \partial p/\partial s \rangle dt.$$

Since  $p(t, s)$  was an arbitrary fixed end point variation, we get

$$p'' - R(v', v)p' = 0,$$

which is Sasaki's second equation.

Thus if the curve  $(p(t), v(t))$  is a geodesic in  $UM$ , then both of Sasaki's equations must be satisfied. Conversely, if these equations are satisfied, then the curve is a critical point of the energy  $E$  for fixed end point variations, and hence a geodesic in  $UM$ . This completes the proof of Sasaki's theorem.

Here are some immediate consequences of Sasaki's theorem.

Suppose  $(p(t), v(t))$  is a constant speed geodesic in  $UM$ . Then:

- 1) The vertical speed  $|v'(t)|$  is constant. Indeed,

$$\langle v, v \rangle = 1 \Rightarrow \langle v, v' \rangle = 0,$$

and hence

$$\partial/\partial t \langle v', v' \rangle = 2 \langle v'', v' \rangle = -2 \langle v', v' \rangle \langle v, v' \rangle = 0,$$

by Sasaki's first equation.

- 2) The horizontal speed  $|p'(t)|$  is also constant. We have

$$\partial/\partial t \langle p', p' \rangle = 2 \langle p'', p' \rangle = 2 \langle R(v', v)p', p' \rangle = 0,$$

by Sasaki's second equation together with the skew-symmetry of the Riemann curvature tensor  $\langle R(\cdot, \cdot)\cdot, \cdot \rangle$  in its last two positions.

- 3) If  $v(t)$  is a parallel vector field along  $p(t)$ , then Sasaki's second equation reduces to the equation  $p'' = 0$  of a geodesic in  $M$ . Conversely, if  $p(t)$  is a geodesic in  $M$  and  $v(t)$  a parallel unit vector field along it, then Sasaki's two equations are clearly satisfied, so  $(p(t), v(t))$  must be a geodesic in  $UM$ . But there will also be geodesics  $(p(t), v(t))$  in  $UM$  for which  $p(t)$  is a geodesic in  $M$ , while  $v(t)$  is *not* parallel along  $p(t)$ .