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The formulas for curvature, torsion and writhe are as follows.

$$\text{Curvature} = \kappa = \sqrt{(a^2 - 1)(1 - b^2)}$$

$$\text{Torsion} = \tau = ab$$

$$\text{Writhe} = \rho = \sqrt{a^2 + b^2 - 1}.$$

Consider the 3-dimensional linear space of vector fields

$$aT(t) + bN(t) + cB(t)$$

which can be written as constant coefficient combinations of the Frenet vectors along the helix $p(t)$. Covariant differentiation along the helix maps this linear space to itself according to the Frenet formulas.

We've already noted in the introduction that the instantaneous axis vector $U = \tau T - \kappa B$ satisfies $U' = 0$.

Consider the vectors N and $V = (\kappa/\rho)T + (\tau/\rho)B$, which form an orthonormal basis for the orthogonal complement of U . Note that

$$N' = -\kappa T - \tau B = -\rho V, \quad \text{and}$$

$$V' = (\kappa/\rho)T' + (\tau/\rho)B' = (\kappa/\rho)(\kappa N) + (\tau/\rho)(\tau N) = \rho N.$$

Thus, covariant differentiation along the helix kills the instantaneous axis vector and takes the orthogonal 2-plane to itself by a 90° rotation, followed by multiplication by the writhe.

4. SASAKI'S EQUATIONS

Let M be any Riemannian manifold, and UM its unit tangent bundle with the Riemannian metric described in section 1.

THEOREM (Sasaki [Sa], 1958). *The curve $(p(t), v(t))$ in UM is a constant speed geodesic there if and only if both of the following equations hold:*

$$1) \quad v'' = -\langle v', v' \rangle v$$

$$2) \quad p'' = R(v', v)p'.$$

Here, primes denote ordinary derivatives with respect to t when applied to functions, and covariant derivatives along the path $p(t)$ when applied to vector fields. For example, the first prime in p'' represents ordinary differentiation, the second, covariant differentiation. The symbol R denotes the Riemann curvature transformation

$$R: TM_p \times TM_p \rightarrow \text{Hom}(TM_p, TM_p).$$

We give a quick proof of Sasaki's theorem, and refer the reader interested in further details both to Sasaki's original paper and to a brief treatment of his result in [Ba-Br-Bu, pages 37-39].

First note that the energy of the curve $(p(t), v(t))$ in UM is given by

$$E = 1/2 \int_0^1 \langle p', p' \rangle dt + 1/2 \int_0^1 \langle v', v' \rangle dt .$$

This curve is a geodesic in UM precisely when it is a critical point of E for fixed end point variations. These include variations which fix all the foot points $p(t)$, that is, fixed end point variations of the second integral. This second integral equals the energy of the curve $u(t)$, lying in the unit sphere in the tangent space to M at $p(0)$, obtained by parallel translating $v(t)$ backwards along $p(t)$ to $p(0)$. Hence the curve $u(t)$ is a geodesic, that is, a great circle arc, in this unit sphere.

Because $u(t)$ is a unit vector field, $\langle u, u \rangle = 1$. Differentiating twice, $\langle u'', u \rangle + \langle u', u' \rangle = 0$. Because $u(t)$ runs at constant speed along a great circle, u'' is parallel to u . Hence $u'' = -\langle u', u' \rangle u$. Parallel translating this equation back out along $p(t)$, we get Sasaki's first equation.

To get Sasaki's second equation, consider a fixed end point variation $(p(t, s), v(t, s))$ of the curve $(p(t), v(t))$ in UM . Suppose this curve is a critical point of the energy E for such variations. Then

$$0 = dE/ds = 1/2 \int_0^1 \partial/\partial s \langle p', p' \rangle dt + 1/2 \int_0^1 \partial/\partial s \langle v', v' \rangle dt .$$

The first integrand is processed by differentiating with respect to s , then interchanging the order of the t and s differentiations, and finally setting up for integration by parts, yielding

$$\partial/\partial t \langle \partial p/\partial s, p' \rangle - \langle \partial p/\partial s, p'' \rangle .$$

The second integrand is processed similarly, except that the Riemann curvature transformation appears as a penalty for interchanging the order of the t and s differentiations, since this time both are covariant. We get

$$\partial/\partial t \langle \partial v/\partial s, v' \rangle - \langle \partial v/\partial s, v'' \rangle + \langle R(\partial p/\partial s, p')v, v' \rangle .$$

Integrating these two expressions with respect to t , as required, the leading term of each drops out because the variation is fixed end point. Furthermore, the second term of the second expression is dead zero: since $\langle v, v \rangle = 1$,

$\partial v/\partial s$ is orthogonal to v , while by Sasaki's first equation, v'' is parallel to v . We get

$$0 = \int_0^1 \langle \partial p/\partial s, p'' \rangle - \langle R(\partial p/\partial s, p')v, v' \rangle dt.$$

Capitalizing on the symmetries of the curvature, we rewrite this as

$$0 = \int_0^1 \langle p'' - R(v', v)p', \partial p/\partial s \rangle dt.$$

Since $p(t, s)$ was an arbitrary fixed end point variation, we get

$$p'' - R(v', v)p' = 0,$$

which is Sasaki's second equation.

Thus if the curve $(p(t), v(t))$ is a geodesic in UM , then both of Sasaki's equations must be satisfied. Conversely, if these equations are satisfied, then the curve is a critical point of the energy E for fixed end point variations, and hence a geodesic in UM . This completes the proof of Sasaki's theorem.

Here are some immediate consequences of Sasaki's theorem.

Suppose $(p(t), v(t))$ is a constant speed geodesic in UM . Then:

- 1) The vertical speed $|v'(t)|$ is constant. Indeed,

$$\langle v, v \rangle = 1 \Rightarrow \langle v, v' \rangle = 0,$$

and hence

$$\partial/\partial t \langle v', v' \rangle = 2 \langle v'', v' \rangle = -2 \langle v', v' \rangle \langle v, v' \rangle = 0,$$

by Sasaki's first equation.

- 2) The horizontal speed $|p'(t)|$ is also constant. We have

$$\partial/\partial t \langle p', p' \rangle = 2 \langle p'', p' \rangle = 2 \langle R(v', v)p', p' \rangle = 0,$$

by Sasaki's second equation together with the skew-symmetry of the Riemann curvature tensor $\langle R(\cdot, \cdot)\cdot, \cdot \rangle$ in its last two positions.

- 3) If $v(t)$ is a parallel vector field along $p(t)$, then Sasaki's second equation reduces to the equation $p'' = 0$ of a geodesic in M . Conversely, if $p(t)$ is a geodesic in M and $v(t)$ a parallel unit vector field along it, then Sasaki's two equations are clearly satisfied, so $(p(t), v(t))$ must be a geodesic in UM . But there will also be geodesics $(p(t), v(t))$ in UM for which $p(t)$ is a geodesic in M , while $v(t)$ is *not* parallel along $p(t)$.