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The formulas for curvature, torsion and writhe are as follows.

$$\text{Curvature} = \kappa = \sqrt{(a^2 - 1)(1 - b^2)}$$

$$\text{Torsion} = \tau = ab$$

$$\text{Writhe} = \rho = \sqrt{a^2 + b^2 - 1}.$$

Consider the 3-dimensional linear space of vector fields

$$aT(t) + bN(t) + cB(t)$$

which can be written as constant coefficient combinations of the Frenet vectors along the helix  $p(t)$ . Covariant differentiation along the helix maps this linear space to itself according to the Frenet formulas.

We've already noted in the introduction that the instantaneous axis vector  $U = \tau T - \kappa B$  satisfies  $U' = 0$ .

Consider the vectors  $N$  and  $V = (\kappa/\rho)T + (\tau/\rho)B$ , which form an orthonormal basis for the orthogonal complement of  $U$ . Note that

$$N' = -\kappa T - \tau B = -\rho V, \quad \text{and}$$

$$V' = (\kappa/\rho)T' + (\tau/\rho)B' = (\kappa/\rho)(\kappa N) + (\tau/\rho)(\tau N) = \rho N.$$

Thus, covariant differentiation along the helix kills the instantaneous axis vector and takes the orthogonal 2-plane to itself by a  $90^\circ$  rotation, followed by multiplication by the writhe.

#### 4. SASAKI'S EQUATIONS

Let  $M$  be any Riemannian manifold, and  $UM$  its unit tangent bundle with the Riemannian metric described in section 1.

**THEOREM** (Sasaki [Sa], 1958). *The curve  $(p(t), v(t))$  in  $UM$  is a constant speed geodesic there if and only if both of the following equations hold:*

$$1) \quad v'' = -\langle v', v' \rangle v$$

$$2) \quad p'' = R(v', v)p'.$$

Here, primes denote ordinary derivatives with respect to  $t$  when applied to functions, and covariant derivatives along the path  $p(t)$  when applied to vector fields. For example, the first prime in  $p''$  represents ordinary differentiation, the second, covariant differentiation. The symbol  $R$  denotes the Riemann curvature transformation

$$R: TM_p \times TM_p \rightarrow \text{Hom}(TM_p, TM_p).$$

We give a quick proof of Sasaki's theorem, and refer the reader interested in further details both to Sasaki's original paper and to a brief treatment of his result in [Ba-Br-Bu, pages 37-39].

First note that the energy of the curve  $(p(t), v(t))$  in  $UM$  is given by

$$E = 1/2 \int_0^1 \langle p', p' \rangle dt + 1/2 \int_0^1 \langle v', v' \rangle dt .$$

This curve is a geodesic in  $UM$  precisely when it is a critical point of  $E$  for fixed end point variations. These include variations which fix all the foot points  $p(t)$ , that is, fixed end point variations of the second integral. This second integral equals the energy of the curve  $u(t)$ , lying in the unit sphere in the tangent space to  $M$  at  $p(0)$ , obtained by parallel translating  $v(t)$  backwards along  $p(t)$  to  $p(0)$ . Hence the curve  $u(t)$  is a geodesic, that is, a great circle arc, in this unit sphere.

Because  $u(t)$  is a unit vector field,  $\langle u, u \rangle = 1$ . Differentiating twice,  $\langle u'', u \rangle + \langle u', u' \rangle = 0$ . Because  $u(t)$  runs at constant speed along a great circle,  $u''$  is parallel to  $u$ . Hence  $u'' = -\langle u', u' \rangle u$ . Parallel translating this equation back out along  $p(t)$ , we get Sasaki's first equation.

To get Sasaki's second equation, consider a fixed end point variation  $(p(t, s), v(t, s))$  of the curve  $(p(t), v(t))$  in  $UM$ . Suppose this curve is a critical point of the energy  $E$  for such variations. Then

$$0 = dE/ds = 1/2 \int_0^1 \partial/\partial s \langle p', p' \rangle dt + 1/2 \int_0^1 \partial/\partial s \langle v', v' \rangle dt .$$

The first integrand is processed by differentiating with respect to  $s$ , then interchanging the order of the  $t$  and  $s$  differentiations, and finally setting up for integration by parts, yielding

$$\partial/\partial t \langle \partial p/\partial s, p' \rangle - \langle \partial p/\partial s, p'' \rangle .$$

The second integrand is processed similarly, except that the Riemann curvature transformation appears as a penalty for interchanging the order of the  $t$  and  $s$  differentiations, since this time both are covariant. We get

$$\partial/\partial t \langle \partial v/\partial s, v' \rangle - \langle \partial v/\partial s, v'' \rangle + \langle R(\partial p/\partial s, p')v, v' \rangle .$$

Integrating these two expressions with respect to  $t$ , as required, the leading term of each drops out because the variation is fixed end point. Furthermore, the second term of the second expression is dead zero: since  $\langle v, v \rangle = 1$ ,

$\partial v/\partial s$  is orthogonal to  $v$ , while by Sasaki's first equation,  $v''$  is parallel to  $v$ . We get

$$0 = \int_0^1 \langle \partial p/\partial s, p'' \rangle - \langle R(\partial p/\partial s, p')v, v' \rangle dt.$$

Capitalizing on the symmetries of the curvature, we rewrite this as

$$0 = \int_0^1 \langle p'' - R(v', v)p', \partial p/\partial s \rangle dt.$$

Since  $p(t, s)$  was an arbitrary fixed end point variation, we get

$$p'' - R(v', v)p' = 0,$$

which is Sasaki's second equation.

Thus if the curve  $(p(t), v(t))$  is a geodesic in  $UM$ , then both of Sasaki's equations must be satisfied. Conversely, if these equations are satisfied, then the curve is a critical point of the energy  $E$  for fixed end point variations, and hence a geodesic in  $UM$ . This completes the proof of Sasaki's theorem.

Here are some immediate consequences of Sasaki's theorem.

Suppose  $(p(t), v(t))$  is a constant speed geodesic in  $UM$ . Then:

- 1) The vertical speed  $|v'(t)|$  is constant. Indeed,

$$\langle v, v \rangle = 1 \Rightarrow \langle v, v' \rangle = 0,$$

and hence

$$\partial/\partial t \langle v', v' \rangle = 2 \langle v'', v' \rangle = -2 \langle v', v' \rangle \langle v, v' \rangle = 0,$$

by Sasaki's first equation.

- 2) The horizontal speed  $|p'(t)|$  is also constant. We have

$$\partial/\partial t \langle p', p' \rangle = 2 \langle p'', p' \rangle = 2 \langle R(v', v)p', p' \rangle = 0,$$

by Sasaki's second equation together with the skew-symmetry of the Riemann curvature tensor  $\langle R(\cdot, \cdot)\cdot, \cdot \rangle$  in its last two positions.

- 3) If  $v(t)$  is a parallel vector field along  $p(t)$ , then Sasaki's second equation reduces to the equation  $p'' = 0$  of a geodesic in  $M$ . Conversely, if  $p(t)$  is a geodesic in  $M$  and  $v(t)$  a parallel unit vector field along it, then Sasaki's two equations are clearly satisfied, so  $(p(t), v(t))$  must be a geodesic in  $UM$ . But there will also be geodesics  $(p(t), v(t))$  in  $UM$  for which  $p(t)$  is a geodesic in  $M$ , while  $v(t)$  is *not* parallel along  $p(t)$ .