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### 3. HELICES IN $S^3$

A spherical helix in  $S^3$  is a curve  $p(t)$  of constant geodesic curvature and torsion. As in  $R^3$ , two spherical helices of the same curvature and torsion are congruent.

If the curvature is nonzero, then we can define a Frenet frame  $T(t)$ ,  $N(t)$ ,  $B(t)$  along  $p(t)$  in the usual way, and get the Frenet equations:

$$T' = \kappa N, \quad N' = -\kappa T - \tau B, \quad B' = \tau N.$$

Here we assume that  $t$  is an arc length parameter along  $p(t)$ , and use primes ' to denote covariant differentiation of vector fields along this path.

A model helix in  $S^3$  is given by

$$p(t) = (\cos \alpha \cos at, \cos \alpha \sin at, \sin \alpha \cos bt, \sin \alpha \sin bt).$$

Here  $\alpha$  ranges between 0 and  $\pi/2$  and determines the shape of the flat torus

$$x_1^2 + x_2^2 = \cos^2 \alpha, \quad x_3^2 + x_4^2 = \sin^2 \alpha,$$

on which the helix  $p(t)$  lies. We take the numbers  $a$  and  $b$  to be  $\geq 0$ , and require that

$$a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = 1,$$

so that the helix will be traversed at unit speed. Every spherical helix in  $S^3$  is congruent to one of these models.

Next, we give formulas for the curvature  $\kappa$ , torsion  $\tau$ , and writhe  $\rho = \sqrt{\kappa^2 + \tau^2}$  of the model helix  $p(t)$  in terms of the descriptive parameters  $\alpha$ ,  $a$  and  $b$ . These formulas are given as general information only, and will not be used here.

We first record two simple inequalities which follow from the equality  $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = 1$ .

Note that  $a = 1$  and  $b = 1$  satisfies this equation. So if one of these quantities increases above 1, the other must decrease below 1. Arranging matters so that  $a$  is the larger of the two, we will then have

$$(a^2 - 1)(1 - b^2) \geq 0.$$

In addition,

$$a^2 + b^2 \geq a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = 1,$$

so we have

$$a^2 + b^2 - 1 \geq 0.$$

The formulas for curvature, torsion and writhe are as follows.

$$\begin{aligned}\text{Curvature} &= \kappa = \sqrt{(a^2 - 1)(1 - b^2)} \\ \text{Torsion} &= \tau = ab \\ \text{Writhe} &= \rho = \sqrt{a^2 + b^2 - 1}.\end{aligned}$$

Consider the 3-dimensional linear space of vector fields

$$aT(t) + bN(t) + cB(t)$$

which can be written as constant coefficient combinations of the Frenet vectors along the helix  $p(t)$ . Covariant differentiation along the helix maps this linear space to itself according to the Frenet formulas.

We've already noted in the introduction that the instantaneous axis vector  $U = \tau T - \kappa B$  satisfies  $U' = 0$ .

Consider the vectors  $N$  and  $V = (\kappa/\rho)T + (\tau/\rho)B$ , which form an orthonormal basis for the orthogonal complement of  $U$ . Note that

$$\begin{aligned}N' &= -\kappa T - \tau B = -\rho V, \quad \text{and} \\ V' &= (\kappa/\rho)T' + (\tau/\rho)B' = (\kappa/\rho)(\kappa N) + (\tau/\rho)(\tau N) = \rho N.\end{aligned}$$

Thus, covariant differentiation along the helix kills the instantaneous axis vector and takes the orthogonal 2-plane to itself by a  $90^\circ$  rotation, followed by multiplication by the writhe.

#### 4. SASAKI'S EQUATIONS

Let  $M$  be any Riemannian manifold, and  $UM$  its unit tangent bundle with the Riemannian metric described in section 1.

**THEOREM** (Sasaki [Sa], 1958). *The curve  $(p(t), v(t))$  in  $UM$  is a constant speed geodesic there if and only if both of the following equations hold:*

- 1)  $v'' = -\langle v', v' \rangle v$
- 2)  $p'' = R(v', v)p'$ .

Here, primes denote ordinary derivatives with respect to  $t$  when applied to functions, and covariant derivatives along the path  $p(t)$  when applied to vector fields. For example, the first prime in  $p''$  represents ordinary differentiation, the second, covariant differentiation. The symbol  $R$  denotes the Riemann curvature transformation

$$R: TM_p \times TM_p \rightarrow \text{Hom}(TM_p, TM_p).$$