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Compute the greatest common divisor  $P_n$  of the (univariate) polynomials in  $G_n$ . Find a zero  $a_n \in R$  of  $P_n$ . If  $P_n$  has no zero in  $R$ , then  $Z(J) = \emptyset$ .

Let  $k \in \{1, \dots, n-1\}$ . Suppose that  $a_{k+1}, \dots, a_n \in R$  have already been found. Let  $G_k(a_{k+1}, \dots, a_n) \subseteq R[X_k]$  be the set of polynomials in one variable  $X_k$  obtained from  $G_k$  by substituting everywhere  $a_j$  for  $X_j$ ,  $k+1 \leq j \leq n$ .

Compute the greatest common divisor  $P_k$  of the polynomials in  $G_k(a_{k+1}, \dots, a_n)$ . Find a zero  $a_k \in R$  of  $P_k$ . If  $P_k$  has no zero in  $R$ , we have to go back to  $G_n$  and to find another sequence  $a'_n, \dots, a'_{k+1}$ .

If we obtain  $(a_1, \dots, a_n)$  by this algorithm, it is an element of  $Z(J)$ . By 4.3. all elements of  $Z(J)$  can be computed in this way.

Suppose that  $Z_K(J)$  is finite (i.e.  $\mathbf{N}^n - \mathcal{D}(G)$  is finite) and that we are able to solve univariate polynomial equations in  $R$  (which is the case for  $R = \mathbf{Z}$ ). Then the algorithm above yields  $Z(J)$  in a finite number of steps.

4.5. *Example.* Let  $F$  be the subset

$$\begin{aligned} & \{2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 - 3X_2^2 - 5X_2X_3 - 2X_3 + 41, \\ & 4X_1^4 + 6X_1^3X_2X_3 - 2X_1X_2^2 + 10X_1 + 3X_2^2 + 5X_2X_3 + 2X_3^3 - 11X_3^2 + 19X_3 + 25, \\ & 6X_2^2 + 10X_2X_3 + 2X_3^3 - 11X_3^2 + 21X_3 - 40\} \quad \text{of} \quad \mathbf{Z}[X_1, X_2, X_3]. \end{aligned}$$

By the algorithm 3.6. we get a Gröbner basis  $G$  of the ideal generated by  $F$ :

$$\begin{aligned} G = & \{2X_3^3 - 11X_3^2 + 17X_3 - 6, \\ & 3X_2^2 + 5X_2X_3 + 2X_3 - 17, \\ & 2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 + 24\}. \end{aligned}$$

Now  $Z(G_3) = \{2, 3\}$ ,  $Z(G_2(2)) = \{1\}$ ,  $Z(G_2(3)) = \emptyset$  and  $Z(G_1(1, 2)) = \{-2\}$ . So  $Z(F) = \{(-2, 1, 2)\}$ .

## 5. APPLICATION TO A GEOMETRIC PROBLEM

5.1. For  $P \in R[X]$  let  $\tilde{P}$  be the homogenization of  $P$  by a further variable  $X_{n+1}$ . For an ideal  $J \leq R[X]$  we write  $\tilde{J}$  for the ideal generated by  $\{\tilde{P} \mid P \in J\}$  in  $R[X_1, \dots, X_{n+1}]$ .

**PROPOSITION.** *Let  $G$  be a Gröbner basis of  $J$  with respect to the graded inverse lexicographic ordering (see 2.1.). Then  $\tilde{G} := \{\tilde{P} \mid P \in G\}$  is a Gröbner basis of  $\tilde{J}$ .*

*Proof.* Since we consider the graded inverse lexicographic ordering, we have for all  $P \in R[X] - \{0\}$ :  $\text{in}(P) = \text{in}(\tilde{P})$ . Hence  $\langle \text{in}(\tilde{J}) \rangle = \langle \text{in}(J) \rangle = \langle \text{in}(G) \rangle = \langle \text{in}(\tilde{G}) \rangle$ .

5.2. *Example.* Let  $R$  be a field. Consider the “twisted cubic”

$$Z := \{(t, t^2, t^3) \mid t \in R\} \subseteq R^3.$$

Then

$$J := \langle X_1^3 - X_3, X_1^2 - X_2 \rangle \leqslant R[X_1, X_2, X_3]$$

is the ideal of polynomials vanishing on  $Z$ .

Recall that the set of zeroes of  $\tilde{J}$  in the projective space  $\mathbf{P}_3(R)$  is the closure (with respect to the Zariski topology) of  $Z$ .

The polynomials  $X_1^3 - X_3X_4^2$  and  $X_1^2 - X_2X_4$  do *not* generate the ideal  $\tilde{J} \leqslant R[X_1, X_2, X_3, X_4]$ .

By 3.6.  $G := \{X_1^2 - X_2, X_1X_2 - X_3, X_2^2 - X_1X_3\}$  is a Gröbner basis of  $J$  with respect to the graded inverse lexicographic ordering. Hence  $\tilde{J}$  is generated by  $\{X_1^2 - X_2X_4, X_1X_2 - X_3X_4, X_2^2 - X_1X_3X_4\}$ .

## REFERENCES

- [Ba] BAYER, D. An Introduction to the Division Algorithm. Preprint 1985.
- [B] BUCHBERGER, B. Gröbner Bases: An Algorithmic Method in Polynomial Ideal Theory. In: Bose, N. (ed.), *Multidimensional Systems Theory*, pp. 184-232. Reidel Publ. Comp., Dordrecht 1985.
- [E] ELIAHOU, S. Minimal Syzygies of Monomial Ideals and Gröbner Bases. Preprint 1987.
- [K1] KANDRI-RODY, A. and D. KAPUR. Computing the Gröbner Basis of an Ideal in Polynomial Rings over the Integers. In: *Proceedings of Third MACSYMA Users Conference*. Schenectady, New York, 1984, pp. 436-451.
- [K2] KANDRI-RODY, A. and D. KAPUR. *An Algorithm for Computing the Gröbner Basis of a Polynomial Ideal over a Euclidian Ring*. Report No. 84CRD045, General Electric Research and Development Center, Schenectady, New York, 1984.