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Compute the greatest common divisor P_n of the (univariate) polynomials in G_n. Find a zero $a_n \in R$ of P_n. If P_n has no zero in R, then $Z(J) = \emptyset$.

Let $k \in \{1, ..., n-1\}$. Suppose that $a_{k+1}, ..., a_n \in R$ have already been found. Let $G_k(a_{k+1}, ..., a_n) \subseteq R[X_k]$ be the set of polynomials in one variable X_k obtained from G_k by substituting everywhere a_j for X_j, $k+1 \leqslant j \leqslant n$.

Compute the greatest common divisor P_k of the polynomials in $G_k(a_{k+1}, ..., a_n)$. Find a zero $a_k \in R$ of P_k. If P_k has no zero in R, we have to go back to G_n and to find another sequence $a'_n, ..., a'_{k+1}$.

If we obtain $(a_1, ..., a_n)$ by this algorithm, it is an element of $Z(J)$. By 4.3. all elements of $Z(J)$ can be computed in this way.

Suppose that $Z_K(J)$ is finite (i.e. $\mathbf{N}^n - \mathscr{D}(G)$ is finite) and that we are able to solve univariate polynomial equations in R (which is the case for $R = \mathbf{Z}$). Then the algorithm above yields $Z(J)$ in a finite number of steps.

4.5. *Example.* Let F be the subset

$$\{2X_1^4+3X_1^3X_2X_3-X_1X_2^2+5X_1-3X_2^2-5X_2X_3-2X_3+41,$$
$$4X_1^4+6X_1^3X_2X_3-2X_1X_2^2+10X_1+3X_2^2+5X_2X_3+2X_3^3-11X_3^2+19X_3+25,$$
$$6X_2^2+10X_2X_3+2X_3^3-11X_3^2+21X_3-40\} \quad \text{of} \quad \mathbf{Z}[X_1, X_2, X_3].$$

By the algorithm 3.6. we get a Gröbner basis G of the ideal generated by F:

$$G = \{2X_3^3-11X_3^2+17X_3-6,$$
$$3X_2^2+5X_2X_3+2X_3-17,$$
$$2X_1^4+3X_1^3X_2X_3-X_1X_2^2+5X_1+24\}.$$

Now $Z(G_3) = \{2, 3\}$, $Z(G_2(2)) = \{1\}$, $Z(G_2(3)) = \emptyset$ and $Z(G_1(1, 2)) = \{-2\}$. So $Z(F) = \{(-2, 1, 2)\}$.

5. Application to a Geometric Problem

5.1. For $P \in R[X]$ let $\tilde{P}$ be the homogeneization of P by a further variable X_{n+1}. For an ideal $J \leqslant R[X]$ we write $\tilde{J}$ for the ideal generated by $\{\tilde{P} \mid P \in J\}$ in $R[X_1, ..., X_{n+1}]$.

Proposition. *Let G be a Gröbner basis of J with respect to the graded inverse lexicographic ordering (see 2.1.). Then $\tilde{G} := \{\tilde{P} \mid P \in G\}$ is a Gröbner basis of $\tilde{J}$.*

Proof. Since we consider the graded inverse lexicographic ordering, we have for all $P \in R[X] - \{0\}: \mathrm{in}(P) = \mathrm{in}(\tilde{P})$. Hence $<\mathrm{in}(\tilde{J})> = <\mathrm{in}(J)> = <\mathrm{in}(G)> = <\mathrm{in}(\tilde{G})>$.

5.2. *Example.* Let R be a field. Consider the "twisted cubic"

$$Z := \{(t, t^2, t^3) \mid t \in R\} \subseteq R^3 .$$

Then

$$J := <X_1^3 - X_3, X_1^2 - X_2> \leqslant R[X_1, X_2, X_3]$$

is the ideal of polynomials vanishing on Z.

Recall that the set of zeroes of $\tilde{J}$ in the projective space $\mathbf{P}_3(R)$ is the closure (with respect to the Zariski topology) of Z.

The polynomials $X_1^3 - X_3X_4^2$ and $X_1^2 - X_2X_4$ do *not* generate the ideal $\tilde{J} \leqslant R[X_1, X_2, X_3, X_4]$.

By 3.6. $G := \{X_1^2 - X_2, X_1X_2 - X_3, X_2^2 - X_1X_3\}$ is a Gröbner basis of J with respect to the graded inverse lexicographic ordering. Hence $\tilde{J}$ is generated by $\{X_1^2 - X_2X_4, X_1X_2 - X_3X_4, X_2^2 - X_1X_3X_4\}$.

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