

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 34 (1988)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE THEORY OF GRÖBNER BASES
Autor: Pauer, Franz / Pfeifhofer, Marlene
Kapitel: 5. Application to a Geometric Problem
DOI: <https://doi.org/10.5169/seals-56595>

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Compute the greatest common divisor P_n of the (univariate) polynomials in G_n . Find a zero $a_n \in R$ of P_n . If P_n has no zero in R , then $Z(J) = \emptyset$.

Let $k \in \{1, \dots, n-1\}$. Suppose that $a_{k+1}, \dots, a_n \in R$ have already been found. Let $G_k(a_{k+1}, \dots, a_n) \subseteq R[X_k]$ be the set of polynomials in one variable X_k obtained from G_k by substituting everywhere a_j for X_j , $k+1 \leq j \leq n$.

Compute the greatest common divisor P_k of the polynomials in $G_k(a_{k+1}, \dots, a_n)$. Find a zero $a_k \in R$ of P_k . If P_k has no zero in R , we have to go back to G_n and to find another sequence a'_n, \dots, a'_{k+1} .

If we obtain (a_1, \dots, a_n) by this algorithm, it is an element of $Z(J)$. By 4.3. all elements of $Z(J)$ can be computed in this way.

Suppose that $Z_K(J)$ is finite (i.e. $\mathbf{N}^n - \mathcal{D}(G)$ is finite) and that we are able to solve univariate polynomial equations in R (which is the case for $R = \mathbf{Z}$). Then the algorithm above yields $Z(J)$ in a finite number of steps.

4.5. *Example.* Let F be the subset

$$\begin{aligned} &\{2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 - 3X_2^2 - 5X_2X_3 - 2X_3 + 41, \\ &4X_1^4 + 6X_1^3X_2X_3 - 2X_1X_2^2 + 10X_1 + 3X_2^2 + 5X_2X_3 + 2X_3^3 - 11X_3^2 + 19X_3 + 25, \\ &6X_2^2 + 10X_2X_3 + 2X_3^3 - 11X_3^2 + 21X_3 - 40\} \quad \text{of} \quad \mathbf{Z}[X_1, X_2, X_3]. \end{aligned}$$

By the algorithm 3.6. we get a Gröbner basis G of the ideal generated by F :

$$\begin{aligned} G = &\{2X_3^3 - 11X_3^2 + 17X_3 - 6, \\ &3X_2^2 + 5X_2X_3 + 2X_3 - 17, \\ &2X_1^4 + 3X_1^3X_2X_3 - X_1X_2^2 + 5X_1 + 24\}. \end{aligned}$$

Now $Z(G_3) = \{2, 3\}$, $Z(G_2(2)) = \{1\}$, $Z(G_2(3)) = \emptyset$ and $Z(G_1(1, 2)) = \{-2\}$. So $Z(F) = \{(-2, 1, 2)\}$.

5. APPLICATION TO A GEOMETRIC PROBLEM

5.1. For $P \in R[X]$ let \tilde{P} be the homogeneization of P by a further variable X_{n+1} . For an ideal $J \subseteq R[X]$ we write \tilde{J} for the ideal generated by $\{\tilde{P} \mid P \in J\}$ in $R[X_1, \dots, X_{n+1}]$.

PROPOSITION. Let G be a Gröbner basis of J with respect to the graded inverse lexicographic ordering (see 2.1.). Then $\tilde{G} := \{\tilde{P} \mid P \in G\}$ is a Gröbner basis of \tilde{J} .

Proof. Since we consider the graded inverse lexicographic ordering, we have for all $P \in R[X] - \{0\}$: $\text{in}(P) = \text{in}(\tilde{P})$. Hence $\langle \text{in}(\tilde{J}) \rangle = \langle \text{in}(J) \rangle = \langle \text{in}(G) \rangle = \langle \text{in}(\tilde{G}) \rangle$.

5.2. *Example.* Let R be a field. Consider the “twisted cubic”

$$Z := \{(t, t^2, t^3) \mid t \in R\} \subseteq R^3.$$

Then

$$J := \langle X_1^3 - X_3, X_1^2 - X_2 \rangle \subseteq R[X_1, X_2, X_3]$$

is the ideal of polynomials vanishing on Z .

Recall that the set of zeroes of \tilde{J} in the projective space $\mathbf{P}_3(R)$ is the closure (with respect to the Zariski topology) of Z .

The polynomials $X_1^3 - X_3X_4^2$ and $X_1^2 - X_2X_4$ do *not* generate the ideal $\tilde{J} \subseteq R[X_1, X_2, X_3, X_4]$.

By 3.6. $G := \{X_1^2 - X_2, X_1X_2 - X_3, X_2^2 - X_1X_3\}$ is a Gröbner basis of J with respect to the graded inverse lexicographic ordering. Hence \tilde{J} is generated by $\{X_1^2 - X_2X_4, X_1X_2 - X_3X_4, X_2^2 - X_1X_3X_4\}$.

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