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(2) \Rightarrow (3): trivial.

(3) \Rightarrow (1): By (3) we have $\text{in}(Q) \in \langle \text{in}(F) \rangle$ for every $Q \in J - \{0\}$. Hence $\langle \text{in}(J) \rangle = \langle \text{in}(F) \rangle$.

2.6. COROLLARY. Let F be a Gröbner basis of an ideal $J \leq R[X]$.

1) F generates J .

2) Let $Q \in R[X]$. Then $Q \in J$ iff a rest of Q after dividing by F is zero.

Proof. Obvious.

2.7. Another characterisation of Gröbner bases can be given as follows:

We shall say that a set $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$ of admissible combinations of F (with pairwise different degrees) is an " F -admissible set", if for all α we have $\deg(L_\alpha) = \alpha$ and $\text{lc}(L_\alpha)$ generates the ideal

$${}_R \langle \text{lc}(P) \mid P \in \langle \text{in}(F) \rangle, \deg(P) = \alpha \rangle .$$

Any F -admissible set is R -linearly independent.

If R is a field the condition on $\text{lc}(L_\alpha)$ is superfluous.

PROPOSITION. Let J be an ideal in $R[X]$ containing F . Then the following conditions are equivalent:

- (1) F is a Gröbner basis of J .
- (2) There is an F -admissible set which is a R -basis of J .
- (3) Every F -admissible set is a R -basis of J .

Proof. Let $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$ be a F -admissible set.

(1) \Rightarrow (3): Let Q be an element of $J - \{0\}$. Division of Q by $\{L_{\deg(Q)}\}$, of its rest \bar{Q} by $\{L_{\deg(\bar{Q})}\}$, ... yields in a finite number of steps an expression of Q as R -linear combination of L_α 's.

(3) \Rightarrow (2): trivial.

(2) \Rightarrow (1): Suppose that $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$ is a R -basis of J . For every $Q \in J - \{0\}$ the initial term of $L_{\deg(Q)}$ divides $\text{in}(Q)$, hence $\text{in}(Q) \in \langle \text{in}(F) \rangle$.

3. CONSTRUCTION OF GRÖBNER BASES

3.1. *Definition.* Let P, Q be elements of $R[X]$, let $\alpha, \beta \in \mathbb{N}^n$ and let $a, b \in R$. Then the polynomial

$$S(P, Q) := aX^\alpha P - bX^\beta Q$$

is called a "S(ubtraction)-polynomial of P, Q " iff

$$\alpha + \deg(P) = \beta + \deg(Q) = \min(\mathcal{D}(\{P\}) \cap \mathcal{D}(\{Q\}))$$

and $\text{lc}(P) \cdot a = \text{lc}(Q) \cdot b =$ a least common multiple of $\text{lc}(P)$ and $\text{lc}(Q)$.

3.2. *Example.* Consider the graded lexicographic ordering on \mathbf{N}^2 and

$$P := 6X_1^3X_2 + 1, \quad Q := 8X_1X_2^2 + 3X_1X_2 + X_2 \in \mathbf{Z}[X_1, X_2].$$

Then

$$4X_2P - 3X_1^2Q = -9X_1^3X_2 - 3X_1^2X_2 + 4X_2 \quad \text{and} \quad -4X_2P + 3X_1^2Q$$

are S-polynomials of P, Q .

See figure 5.

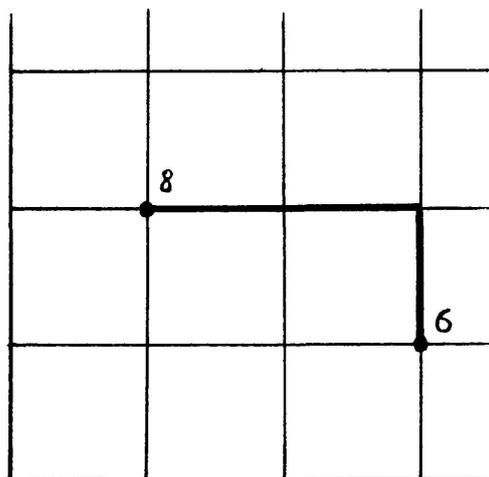


FIGURE 5.

3.3 *Remark.* For $P, Q \in R[X]$, $S(P, Q)$ as defined above is unique up to multiplication by an invertible element of R . Therefore we shall call it "the" S-polynomial of P, Q .

3.4. LEMMA. Let $P_1, \dots, P_k \in R[X]$, $c_1, \dots, c_k \in R$ such that $\deg(P_1) = \dots = \deg(P_k) =: \delta$ but $\deg(\sum_{i=1}^k c_i P_i) \neq \delta$.

Then $\sum_{i=1}^k c_i P_i$ is a R -linear combination of the S-polynomials $S(P_i, P_j)$, $1 \leq i, j \leq k$.

Proof. By induction on k .

Let $l_i := \text{lc}(P_i)$, $1 \leq i \leq k$. Then $\sum_{i=1}^k c_i l_i = 0$.

It is sufficient to prove the existence of $a_{ij}, b_{ij} \in R$ such that

$$\sum_{i=1}^k c_i P_i = \sum_{1 \leq i, j \leq k} (a_{ij} P_i - b_{ij} P_j) \quad \text{and} \quad a_{ij} l_i = b_{ij} l_j, \quad 1 \leq i, j \leq n.$$

For $k = 2$ we have $c_1 P_1 + c_2 P_2 = c_1 P_1 - (-c_2) P_2$ and $c_1 l_1 = (-c_2) l_2$.
 $k = 3$: Let l be a greatest common divisor of l_1, l_2, l_3 . Since $c_2 l_2 = -c_1 l_1 - c_3 l_3$, a greatest common divisor of l_1 and l_3 divides $c_2 l$. Hence there are elements $x_2, x_3 \in R$ such that $c_2 l = x_1 l_1 + x_3 l_3$.

Then $d_1 := (-x_1 l_2 - c_1 l)/l$, $d_2 := (-x_1 l_1)/l$, $d_3 := (x_3 l_2)/l$ are elements of R . Furthermore, we have

$$\begin{aligned} (c_1 + d_1) l_1 &= d_2 l_2 \\ (c_2 + d_2) l_2 &= d_3 l_3 \\ (c_3 + d_3) l_3 &= d_1 l_1 \quad \text{and} \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^3 c_i P_i &= [(c_1 + d_1) P_1 - d_2 P_2] + [(c_2 + d_2) P_2 - d_3 P_3] \\ &\quad + [(c_3 + d_3) P_3 - d_1 P_1]. \end{aligned}$$

$$k > 3: \text{ Let } Q := \sum_{i=3}^k c_i P_i \quad \text{and} \quad m := \sum_{i=3}^k c_i l_i.$$

If $m = 0$, we can apply the induction hypothesis to Q .

If $m \neq 0$, by the $k = 3$ case there are $d_1, d_2, d_3 \in R$ such that

$$\begin{aligned} c_1 P_1 + c_2 P_2 + Q &= [(c_1 + d_1) P_1 - d_2 P_2] + [(c_2 + d_2) P_2 - d_3 Q] \\ &\quad + [(1 + d_3) Q - d_1 P_1] \end{aligned}$$

$$\text{and} \quad (c_1 + d_1) l_1 = d_2 l_2, \quad (c_2 + d_2) l_2 = d_3 m, \quad (1 + d_3) m = d_1 l_1.$$

Therefore, we can apply the induction hypothesis to $(c_2 + d_2) P_2 - \sum_{i=3}^k d_3 c_i P_i$ and to $-d_1 P_1 + \sum_{i=3}^k (1 + d_3) c_i P_i$ and thus terminate the proof.

Remark. If R is a field, the proof is trivial: Let $l_i := \text{lc}(P_i)$ and

$$\begin{aligned} P'_i := (P_i/l_i), \quad 1 \leq i \leq k, \quad \text{then} \quad \sum_{i=1}^k c_i P_i &= c_1 l_1 (P'_1 - P'_2) \\ &\quad + (c_1 l_1 + c_2 l_2) (P'_2 - P'_3) + \dots + \left(\sum_{i=1}^{k-1} c_i l_i \right) (P'_{k-1} - P'_k). \end{aligned}$$

3.5. THEOREM. Let J be an ideal of $R[X]$ generated by a finite subset $F \subseteq R[X] - \{0\}$.

Then the following assertions are equivalent:

- (1) F is a Gröbner basis of J .
- (2) For all $P, Q \in F$ a rest of $S(P, Q)$ after division by F is zero.

Proof.

(1) \Rightarrow (2): Let $P, Q \in F$. Then $S(P, Q)$ and its rest after division by F are elements of J . Therefore, this implication is a special case of proposition 2.5., (1) \Rightarrow (2).

(2) \Rightarrow (1): Let $A \in J - \{0\}$. We have to show that $\text{in}(A) \in \langle \text{in}(F) \rangle$. Since J is generated by F , there are elements $c(\gamma, P) \in R$ such that $A = \sum_{P \in F, \gamma \in \mathbb{N}^n} c(\gamma, P) X^\gamma P$.

Let $\delta := \max \{ \gamma + \deg(P) \mid c(\gamma, P) \neq 0 \}$ and $L := \sum_{\substack{\gamma, P \\ \gamma, \deg(P) = \delta}} c(\gamma, P) X^\gamma P$.

By lemma 1.3. we may assume that δ is minimal, i.e.:

if $A = \sum_{P \in F, \gamma \in \mathbb{N}^n} d(\gamma, P) X^\gamma P$ then $\delta \leq \max \{ \gamma + \deg(P) \mid d(\gamma, P) \neq 0 \}$.

Suppose that $\deg(L) < \delta$. Then the lemma above yields

$$L = \sum_{P, Q \in F, \alpha \in \mathbb{N}^n} a(\alpha, P, Q) X^\alpha S(P, Q), \quad a(\alpha, P, Q) \in R$$

(note that for $\beta, \gamma \in \mathbb{N}^n$ there is an $\alpha \in \mathbb{N}^n$ such that $S(X^\beta P, X^\gamma Q) = X^\alpha S(P, Q)$).

But according to (2) the S -polynomials are admissible combinations of F and clearly the same holds for the $X^\alpha S(P, Q)$'s. Since their degree is smaller than δ , this is a contradiction to the minimality of δ . Hence $\deg(L) = \delta$. But then $\text{in}(A) = \text{in}(L) \in \langle \text{in}(F) \rangle$.

3.6. THEOREM. Let J be the ideal generated by F . Then a Gröbner basis of J can be constructed (in a finite number of steps) by the following algorithm:

$$F_0 := F$$

$$F_{i+1} := F_i \cup (\overline{\{S(P, Q) \mid P, Q \in F_i\}} - \{0\})$$

($\overline{S(P, Q)}$ is a rest of $S(P, Q)$ after division by F_i). If $F_i = F_{i+1}$, then F_i is a Gröbner basis of J .

Proof. By the preceding theorem we only have to show that there is a $k \in \mathbf{N}$ such that $F_k = F_{k+1}$.

If $F_i \subset F_{i+1}$ then $\langle \text{in}(F_i) \rangle \subset \langle \text{in}(F_{i+1}) \rangle$. Since the strictly ascending sequence $\langle \text{in}(F_0) \rangle \subset \langle \text{in}(F_1) \rangle \subset \dots$ must be finite, there is a $k \in \mathbf{N}$ with $F_k = F_{k+1}$.

3.7. *Example.* Consider the graded lexicographic ordering on \mathbf{N}^2 and

$$F := \{P_1 := 2X_1X_2^2 - X_1, P_2 := 3X_1^2X_2 - X_2\} \subseteq \mathbf{Z}[X_1, X_2].$$

Then

$$\begin{aligned} F_0 = F \quad \text{and} \quad S(P_1, P_2) &= 3X_1P_1 - 2X_2P_2 = -3X_1^2 + 2X_2^2 \\ &= \overline{S(P_1, P_2)} = :P_3. \end{aligned}$$

So

$$F_1 = \{P_1, P_2, P_3\} \quad \text{and} \quad \overline{S(P_1, P_2)}^{F_1} = 0,$$

$$\overline{S(P_1, P_3)}^{F_1} = 4X_2^4 - 3X_1^2 = :P_4, \quad \overline{S(P_2, P_3)}^{F_1} = 2X_2^3 - X_2 = :P_5.$$

Therefore $F_2 = \{P_1, P_2, P_3, P_4, P_5\}$ and all rests after division by F_2 of S -polynomials are 0. Hence F_2 is a Gröbner basis of the ideal generated by F .

See figure 6.

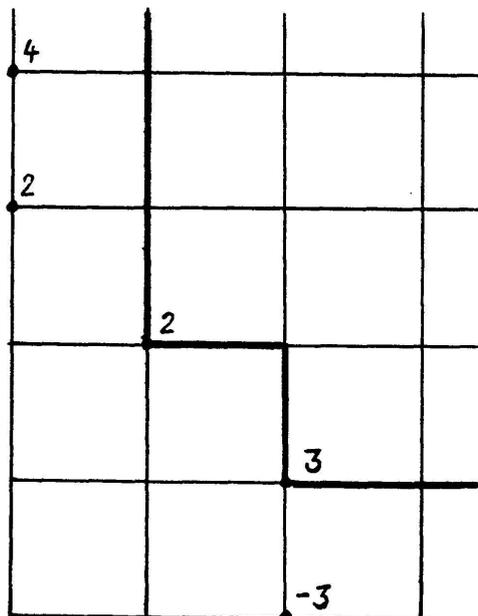


FIGURE 6.

3.8. *Remark.* Let G be a Gröbner basis of an ideal J . We shall say that G is "simplified" if all $P \in G$ fulfill the following two conditions:

$$\text{lc}(P) \text{ generates the ideal } {}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle$$

and

$$\text{in}(P) \notin \langle \text{in}(G - \{P\}) \rangle .$$

It is easy to see that the elements of a simplified Gröbner basis have pairwise different degrees.

If R is a field then G is simplified iff the elements of G have pairwise different degrees and $\deg(G)$ is the set of minimal elements (with respect to the natural partial ordering on \mathbb{N}^n) in $\deg(J)$.

If G is not simplified, then in the following way we can construct (in a finite number of steps) a simplified Gröbner basis of J :

For every $P \in G$ choose an admissible combination P' of G such that $\deg(P) = \deg(P')$ and $\text{lc}(P')$ generates the ideal

$${}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle .$$

Then $G' := \{P' \mid P \in G\}$ is a Gröbner basis of J , since $\langle \text{in}(J) \rangle = \langle \text{in}(G) \rangle \subseteq \langle \text{in}(G') \rangle \subseteq \langle \text{in}(J) \rangle$.

If there is a $P' \in G'$ with $\text{in}(P') \in \langle \text{in}(G' - \{P'\}) \rangle$, then $G' - \{P'\}$ is a Gröbner basis, since then $\langle \text{in}(G' - \{P'\}) \rangle = \langle \text{in}(G') \rangle = \langle \text{in}(J) \rangle$.

Replace G' by $G' - \{P'\}$. After finitely many eliminations of this kind we obtain a simplified Gröbner basis.

In example 3.7. the Gröbner basis F_2 is not simplified, since $\text{in}(P_2) = -X_2 \text{in}(P_3)$ and $\text{in}(P_4) = 2X_2 \text{in}(P_5)$. $\{P_1, P_3, P_5\}$ is a simplified Gröbner basis of the ideal generated by F_2 .

4. APPLICATION TO SYSTEMS OF ALGEBRAIC EQUATIONS

Let J be an ideal in $R[X]$, generated by a subset $F \neq \{0\}$.

4.1. We may consider F as a system of algebraic equations in n variables. We denote by K an algebraic closure of the quotient field of R .

Let $Z(F)$ (resp. $Z_K(F)$) be the set $\{z \in R^n$ (resp. K^n) $\mid P(z) = 0$ for all $P \in F\}$ of common zeros in R^n (resp. K^n) of the elements of F . Clearly $Z(F) = Z(J)$ and $Z_K(F) = Z_K(J)$.