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If  $R$  is a field, then  $w: \deg(J) \rightarrow R$  is a "weight function".

$$\delta \mapsto 1$$

So the corresponding figure is of the form

|   |   |   |   |   |
|---|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| • | • | • | • | • |
| 1 | 1 | 1 | • | • |
| • | • | • | • | • |
| • | • | • | • | • |
| • | • | • | • | • |

FIGURE 3.

## 2. THE DIVISION ALGORITHM

Let  $F$  be a finite subset of  $R[X] - \{0\}$ .

2.1. *Definition.* An "admissible combination of  $F$ " is an expression of the form  $L := \sum_{\gamma \in \mathbb{N}^n, P \in F} c(\gamma, P) X^\gamma P$ ,  $c(\gamma, P) \in R$ , such that

$$\deg(L) = \max \{ \deg(X^\gamma P) \mid c(\gamma, P) \neq 0 \}.$$

*Example.* Let  $P, Q \in R[X]$  and let  $\alpha, \beta \in \mathbb{N}^n$ . Then  $X^\alpha P - X^\beta Q$  is an admissible combination of  $\{P, Q\}$  iff  $X^\alpha \cdot \text{in}(P) \neq X^\beta \cdot \text{in}(Q)$ .

*Remark.* For every  $Q \in \langle \text{in}(F) \rangle$  there is an admissible combination  $L$  of  $F$  such that  $\text{in}(L) = \text{in}(Q)$ .  $L$  can be calculated in the following way:

Let  $F' := \{P \in F \mid \deg(Q) - \deg(P) \in \mathbb{N}^n\}$ . Then

$$Q \in \langle \text{in}(F') \rangle \quad \text{and} \quad \text{lc}(Q) \in {}_R \langle \text{lc}(P) \mid P \in F' \rangle.$$

For  $P \in F'$  we calculate elements  $c(P) \in R$  such that  $\text{lc}(Q) = \sum_{P \in F'} c(P) \text{lc}(P)$ .

Set  $L := \sum_{P \in F'} c(P) X^{\deg(Q) - \deg(P)} P$ .

*Example:*  $F := \{5X_1 + 1, 3X_2 + 2\}$ ,  $Q := X_1^2 X_2^3$ .

Then  $L = -X_1 X_2^3 (5X_1 + 1) + 2X_1^2 X_2^2 (3X_2 + 2)$ .

2.2. PROPOSITION. Every  $Q \in R[X] - \{0\}$  may be written as  $Q = L + \bar{Q}$  with the following properties:

If  $\text{in}(Q) \notin \langle \text{in}(F) \rangle$ , then  $L = 0$  and  $Q = \bar{Q}$ .

If  $\text{in}(Q) \in \langle \text{in}(F) \rangle$ , then  $L$  is an admissible combination of  $F$  with  $\text{in}(L) = \text{in}(Q)$ , and either  $\bar{Q} = 0$  or  $\text{in}(\bar{Q}) \notin \langle \text{in}(F) \rangle$ .

$L$  and  $\bar{Q}$  can be found in a finite number of steps by the following algorithm:

$$Q_0 := Q;$$

For  $k \in \mathbf{N}$  assume that  $Q_k$  has already been defined. If  $\text{in}(Q_k) \in \langle \text{in}(F) \rangle$ , we define  $Q_{k+1} := Q_k - L_k$ , where  $L_k$  is an admissible combination of  $F$  with  $\text{in}(L_k) = \text{in}(Q_k)$ .

If  $Q_k = 0$  or  $\text{in}(Q_k) \notin \langle \text{in}(F) \rangle$ , then  $L := \sum_{j=0}^{k-1} L_j$  and  $\bar{Q} := Q_k$ .

*Proof.* We only have to show that there is a number  $k \in \mathbf{N}$  such that  $\text{in}(Q_k) \notin \langle \text{in}(F) \rangle$  or  $Q_k = 0$ .

If  $\text{in}(Q_j) \in \langle \text{in}(F) \rangle$ , then  $\deg(Q_j) > \deg(Q_{j+1})$ , so the assertion follows from the lemma 1.3.

2.3. Definition. The algorithm above is called "division by  $F$ ", the polynomial  $\bar{Q}$  (or, more precisely,  $\bar{Q}^F$ ) is "a rest of  $Q$  after division by  $F$ ".

*Remarks.*

- 1) Even if the strict ordering  $<$  is fixed,  $\bar{Q}$  depends on the choice of the  $L_k$ 's in the algorithm. Hence  $\bar{Q}$  is in general not uniquely determined by  $Q$  and  $F$ .
- 2) If a rest of  $Q$  after division by  $F$  is zero, then  $Q$  belongs to the ideal generated by  $F$ . In general the inverse is not true.

2.4. Example. Consider the graded lexicographic ordering and

$$P_1 := 2X_1^2 + X_1X_2, \quad P_2 := 3X_2^2 + X_1 \in \mathbf{Z}[X_1, X_2].$$

Let  $F$  be  $\{P_1, P_2\}$  and let  $Q := 2X_1^3X_2^3 + X_1X_2$ . Then  $Q_0 = Q$ .

$$L_0 := 2X_1^3X_2P_2 - 2X_1X_2^3P_1,$$

$$Q_1 := Q_0 - L_0 = -2X_1^2X_2^4 + 2X_1^4X_2 + X_1X_2.$$

$$L_1 := -2X_1^2X_2^2P_2 + 2X_2^4P_1,$$

$$Q_2 := Q_1 - L_1 = -2X_1X_2^5 + 2X_1^4X_2 + 2X_1^3X_2^2 + X_1X_2.$$

Now in  $(Q_2) \notin \langle \text{in}(F) \rangle$ , therefore  $Q = L_0 + L_1 + Q_2$ .

See figure 4.

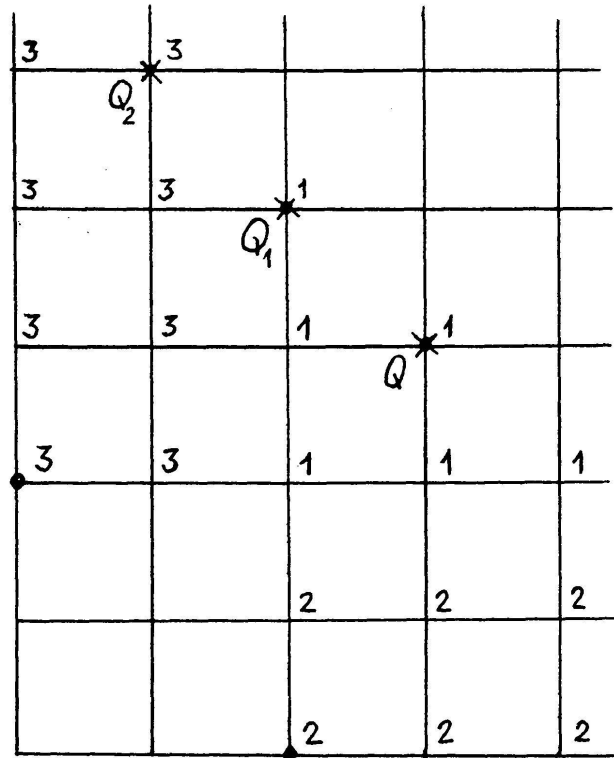


FIGURE 4.

But if we choose  $L'_0 := X_1 X_2^3 P_1$ , then

$$Q'_1 := Q_0 - L'_0 = -X_1^2 X_2^4 + X_1 X_2,$$

$$L'_1 := -X_1^2 X_2^2 P_2 + X_2^4 P_1$$

$$Q'_2 := Q'_1 - L'_1 = -X_1 X_2^5 + X_1^3 X_2^2 + X_1 X_2,$$

$$\text{therefore } Q = L'_0 + L'_1 + Q'_2.$$

So  $Q_2$  and  $Q'_2$  are rests of  $Q$  after division by  $F$  and  $Q_2 \neq Q'_2$ .

2.5. PROPOSITION. Let  $J$  be an ideal in  $R[X]$  containing  $F$ . Then the following conditions are equivalent:

- (1)  $F$  is a Gröbner basis of  $J$ .
- (2) For every  $Q \in J$ , each rest of  $Q$  after division by  $F$  is zero.
- (3) For every  $Q \in J$ , a rest of  $Q$  after division by  $F$  is zero.

*Proof.*

(1)  $\Rightarrow$  (2): Division of  $Q \in J$  by  $F$  yields  $Q = L + \bar{Q}$  with  $\bar{Q} = 0$  or  $\text{in}(\bar{Q}) \notin \langle \text{in}(F) \rangle$ . Now  $L \in J$  and  $Q \in J$  imply  $\bar{Q} \in J$ . Since  $\langle \text{in}(J) \rangle = \langle \text{in}(F) \rangle$ ,  $\bar{Q}$  must be zero.

(2)  $\Rightarrow$  (3): trivial.

(3)  $\Rightarrow$  (1): By (3) we have  $\text{in}(Q) \in \langle \text{in}(F) \rangle$  for every  $Q \in J - \{0\}$ . Hence  $\langle \text{in}(J) \rangle = \langle \text{in}(F) \rangle$ .

2.6. COROLLARY. Let  $F$  be a Gröbner basis of an ideal  $J \leq R[X]$ .

1)  $F$  generates  $J$ .

2) Let  $Q \in R[X]$ . Then  $Q \in J$  iff a rest of  $Q$  after dividing by  $F$  is zero.

*Proof.* Obvious.

2.7. Another characterisation of Gröbner bases can be given as follows:

We shall say that a set  $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$  of admissible combinations of  $F$  (with pairwise different degrees) is an " $F$ -admissible set", if for all  $\alpha$  we have  $\deg(L_\alpha) = \alpha$  and  $\text{lc}(L_\alpha)$  generates the ideal

$${}_R \langle \text{lc}(P) \mid P \in \langle \text{in}(F) \rangle, \deg(P) = \alpha \rangle .$$

Any  $F$ -admissible set is  $R$ -linearly independent.

If  $R$  is a field the condition on  $\text{lc}(L_\alpha)$  is superfluous.

PROPOSITION. Let  $J$  be an ideal in  $R[X]$  containing  $F$ . Then the following conditions are equivalent:

- (1)  $F$  is a Gröbner basis of  $J$ .
- (2) There is an  $F$ -admissible set which is a  $R$ -basis of  $J$ .
- (3) Every  $F$ -admissible set is a  $R$ -basis of  $J$ .

*Proof.* Let  $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$  be a  $F$ -admissible set.

(1)  $\Rightarrow$  (3): Let  $Q$  be an element of  $J - \{0\}$ . Division of  $Q$  by  $\{L_{\deg(Q)}\}$ , of its rest  $\bar{Q}$  by  $\{L_{\deg(\bar{Q})}\}$ , ... yields in a finite number of steps an expression of  $Q$  as  $R$ -linear combination of  $L_\alpha$ 's.

(3)  $\Rightarrow$  (2): trivial.

(2)  $\Rightarrow$  (1): Suppose that  $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$  is a  $R$ -basis of  $J$ . For every  $Q \in J - \{0\}$  the initial term of  $L_{\deg(Q)}$  divides  $\text{in}(Q)$ , hence  $\text{in}(Q) \in \langle \text{in}(F) \rangle$ .

### 3. CONSTRUCTION OF GRÖBNER BASES

3.1. *Definition.* Let  $P, Q$  be elements of  $R[X]$ , let  $\alpha, \beta \in \mathbb{N}^n$  and let  $a, b \in R$ . Then the polynomial