Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 33 (1987)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SKETCH OF THE EVOLUTION OF (NONCOMMUTATIVE) RING THEORY

Autor: Kleiner, Israel

Kapitel: VII. SOME SUBSEQUENT DEVELOPMENTS

DOI: https://doi.org/10.5169/seals-87895

VII. Some subsequent developments

Cartan's structure theorems of the end of the 19th century can be said to have brought to an end the first phase in the evolution of the structure theory of noncommutative rings (algebras), Wedderburn's theorems of 1907 the second phase, and Artin's results of 1927 the third. Artin's work ushered in the next phase, still with us today, in which his work served as a model, a guide, and an inspiration for various subsequent developments in noncommutative ring theory. Thus in (a) and (b) below (see outline on pp. 2-3) we deal with two basic questions left open in the Wedderburn-Artin structure theorems, namely the nature of division rings and nilpotent rings, respectively. In (c), (d), and (e) we discuss generalizations of the Wedderburn-Artin theorems obtained by weakening or removing one or both of its two conditions (semi-simplicity and the minimum condition); this gives rise to quasi-Frobenius rings, primitive rings, and prime rings, respectively. In (e) we comment on the neighbouring area of representations of rings and algebras which, as Herstein noted (see p. 256), derives from the Wedderburn-Artin structure theorems, and in (f) we touch on a fundamental "new" method—homological algebra. We now very briefly sketch these developments.

(a) Division rings (algebras)

We first want to point out that the study of division rings is coextensive with that of division algebras, since the centre of a division ring is a field, and hence the division ring can be viewed as a division algebra over its centre.

The Wedderburn-Artin structure theorems leave unresolved the nature of division algebras over an arbitrary field. Over the fields **R**, **C**, and a finite field, all finite-dimensional division algebras were known by the end of the first decade of the 20th century (see pp. 250-251). In the next two decades, Dickson, Wedderburn, and others introduced new classes of division algebras, especially the important class of cyclic division algebras.[1]) The major problem tackled and completely solved during the late 1920s and early 1930s was the classification of division algebras over the rationals and, more generally, over an algebraic number field. The result, known as the Albert-Brauer-Hasse-Noether theorem, is that all such division algebras are cyclic. Jacobson [48] called this

---

[1]) An algebra A over a field F is *cyclic* if there exists a nonzero element $u \in A$ such that $u^n \in F$ for some positive integer n, and if there exists a maximal subfield K of A which is invariant under the inner automorphisms induced by u, and such that $1, u, \ldots, u^{n-1}$ form a K-basis of A. See e.g. [3] or [4].

result "one of the most important achievements of algebra and number theory in the 1930s." A central tool in the study of these algebras, invented by Brauer in 1929, is the "Brauer group" of a field. The proof also involved deep arithmetic properties of algebraic number fields. See [3], [80] for details.

The problem of the classification of (even finite-dimensional) division algebras is far from solved. Intense research to understand their structure is going on nowadays, using such high-powered tools as K-theory and étale cohomology. See [46] and the recent books [23] and [30] on the subject.

(b) Nilpotent rings

The Wedderburn-Artin structure theorems deal with algebras and rings which are nilpotent-free (semi-simple). At the end of his paper of 1907 on the structure of algebras Wedderburn notes that "the classification of algebras cannot be carried much further than this till a classification of nilpotent algebras has been found..." Attempts had been made in the past to deal with the problem (see e.g. [42]), but, to this day, the results are fragmentary (see e.g. [52]).

(c) Quasi-Frobenius rings

The class of quasi-Frobenius rings is one of the most interesting classes of non-semi-simple rings. A ring R is *quasi-Frobenius* if R satisfies the minimum condition on (say) right ideals and if $rl(J) = J$ and $lr(L) = L$ for every right ideal J and left ideal L of R, where for any subset S of R, $l(S) = \{x \in R : xS = 0\}$, $r(S) = \{x \in R : Sx = 0\}$.

One can show that a semi-simple ring with minimum condition is quasi-Frobenius, hence quasi-Frobenius rings are generalizations of the rings studied by Artin in his structure theorem. In fact, many of the results which can be proved for semi-simple rings with minimum condition are also true for quasi-Frobenius rings, but not for arbitrary rings with minimum condition. That is one reason for studying quasi-Frobenius rings. A second, and the original, reason derives from the theory of group representations. Frobenius introduced the so-called Frobenius algebras around the turn of this century in that context (see [22]), and Nakayama later (1939-41) generalized these to quasi-Frobenius rings and algebras.

In the theory of group representations, Maschke's theorem is of fundamental importance. It states that if G is a finite group and F a field whose characteristic does not divide the order of G, then the group algebra $F(G)$ is semi-simple (and, of course, satisfies the minimum condition, since G is finite).[1]) This result, Nakayama showed, generalizes as follows: If G is finite, F *any* field, then $F(G)$ is quasi-Frobenius. In fact, the following even more general result holds: If R is any quasi-Frobenius ring, G any finite group, then the group ring $R(G)$ is quasi-Frobenius. This is the representation-theoretic context of quasi-Frobenius rings.

Quasi-Frobenius rings have also been generalized, and these rings with their generalizations form the subject of much current research interest. See [76] for details.

(d) PRIMITIVE RINGS

In the 1940s Jacobson arrived at an important extension of the Wedderburn-Artin structure theorems to rings *without* minimum condition, the so-called semi-primitive and primitive rings. The basic problem was to find the "right" definition of the "radical" of a ring—the previous definition of the radical as the maximal nilpotent ideal not being applicable since the ring no longer satisfied the minimum condition. The problem is not trivial. As Herstein notes [43]:

> The aim of defining the radical [is] to concentrate the bothersome behaviour of a ring in a piece of it such that when this piece [is] removed the resulting ring [is] well enough behaved to permit some delicate dissection.

This Jacobson did extremely well. The proof was, of course, in the fruitful results obtained by employing this radical.[2])

A ring is called *semi-primitive* if its (Jacobson) radical is zero. (For rings with minimum condition the notions of semi-primitivity and semi-simplicity coincide.) The other basic notion defined by Jacobson was that of "primitive ring", which was a generalization of simple ring.[3]) Given these definitions, the structure theorem states that:

(a) a semi-primitive ring is a subdirect sum of primitive rings.

---

[1]) It is also known that $F(G)$ is semi-simple for any group G (not necessarily finite) and a non-countable field F of characteristic zero. It is not known, however, if $F(G)$ is semi-simple for countable F.

[2]) One definition of the Jacobson radical is as the intersection of the maximal left (or right) ideals of the ring. There are many equivalent descriptions. See [46], [48] for details. See also [33] for various other radicals.

[3]) A ring is called (left) *primitive* if it contains a maximal left ideal which contains no nonzero ideals of the ring. Again, there are many equivalent descriptions (see [46]). In the presence of the minimum condition, primitivity and simplicity of a ring are equivalent concepts.

(b) a primitive ring is isomorphic to a "dense ring of linear transformations" of a vector space over a division ring. (A "dense ring of linear transformations" is a certain subring of the ring of linear transformations in an infinite-dimensional vector space over a division ring. For example, the ring of row-finite matrices is such a ring.)

This structure theorem of Jacobson was a far-reaching generalization of the Wedderburn-Artin structure theorem. See e.g. [43], [48] for details.

(e) PRIME RINGS

Quasi-Frobenius rings are non-semi-simple rings. Primitive rings do not satisfy the minimum condition. Thus in each of these two cases one of the two conditions in the rings which Artin studied—semi-simple rings with minimum condition—is eliminated. In the present case of prime rings both semi-simplicity and the minimum condition are replaced by weaker conditions. This provides another very important extension, obtained by Goldie in 1958-60, of the Wedderburn-Artin theorems, to semi-prime (and prime) rings with maximum condition. (As we noted, in the presence of an identity in the ring, the minimum condition implies the maximum condition.)

A ring is *semi-prime* if it has no nonzero nilpotent ideals. (For rings with minimum condition this is equivalent to semi-simplicity.) A ring is *prime* if $AB = 0$ implies $A = 0$ or $B = 0$, where A and B are ideals of the ring. Armed with these definitions, Goldie states the structure theorem as follows:

(a) A ring R is semi-prime with maximum condition if and only if it has a classical quotient ring $Q(R)$ which is a semi-simple ring with minimum condition.[1])

(b) A ring R is prime with maximum condition if and only if R has a quotient ring $Q(R)$ which is simple with minimum condition (i.e., by the Wedderburn-Artin theorem, if and only if R can be "tightly embedded" in an $n \times n$ matrix ring over a division ring—a type of "dense embedding" of R in a finite matrix ring, just as a primitive ring can be "densely embedded" in an "infinite matrix ring", that is, in a ring of linear transformations of an infinite dimensional vector space). See [43], [46] for details.

Goldie's theorem assumes the same all-important role in the study of rings with the ascending chain condition (Noetherian rings) which the Wedderburn-

[1]) (i) The maximum condition is equivalent to the ascending chain condition. Rings, not necessarily commutative, satisfying this condition are often called Noetherian rings. (ii) Goldie demands chain conditions somewhat weaker than the maximum condition. (iii) The "classical quotient ring" of a ring is an extension of the notion of the "field of quotients" of an integral domain.

Artin theorem assumes in the study of rings with the descending chain condition (Artinian rings). In fact, the study of noncommutative Noetherian rings, stimulated by Goldie's work, has been an active area of research ever since. One of the major problems is to be able to "do" algebraic geometry in noncommutative Noetherian rings, and thus to extend such notions as localization, divisibility, and torsion to the noncommutative case. See for example [20], [32].

(f) Representations of rings and algebras

The theory of group representations—a very important device in the study of finite groups—was developed by Burnside, Frobenius, and Molien towards the end of the 19th century (see [40]). It was soon found that the group algebra of the group over an appropriate field was an important tool in the study of the representations of the group. (The group algebra is semi-simple and hence the structure theorems of Cartan and Wedderburn can be applied.) In fact, the study of the representations of a group can easily be shown to be reducible to the study of the representations of the group algebra of the given group. Thus attention turned to representations of algebras and then (after the Artin structure theorem) to those of rings. In a fundamental paper in 1929, Noether had shown the conceptual advantages of a shift of focus—from representations of algebras and rings to modules over these algebras and rings, respectively. This is, in large measure, the point of view taken today.

Major departures following the work of Noether were taken by Brauer and Köthe in the 1930s. The major problem is the decomposition of modules over rings into "simple" components. What these simple components are, which modules decompose, which rings give rise to a decomposition of all modules over these rings, are some of the basic questions asked. New classes of rings arose from these studies, among them quasi-Frobenius rings (as we have seen) and uniserial rings. This is a very active field of current interest. See [34], [72], [76] for some of the recent work, and [22] for an account of the "classical" theory.

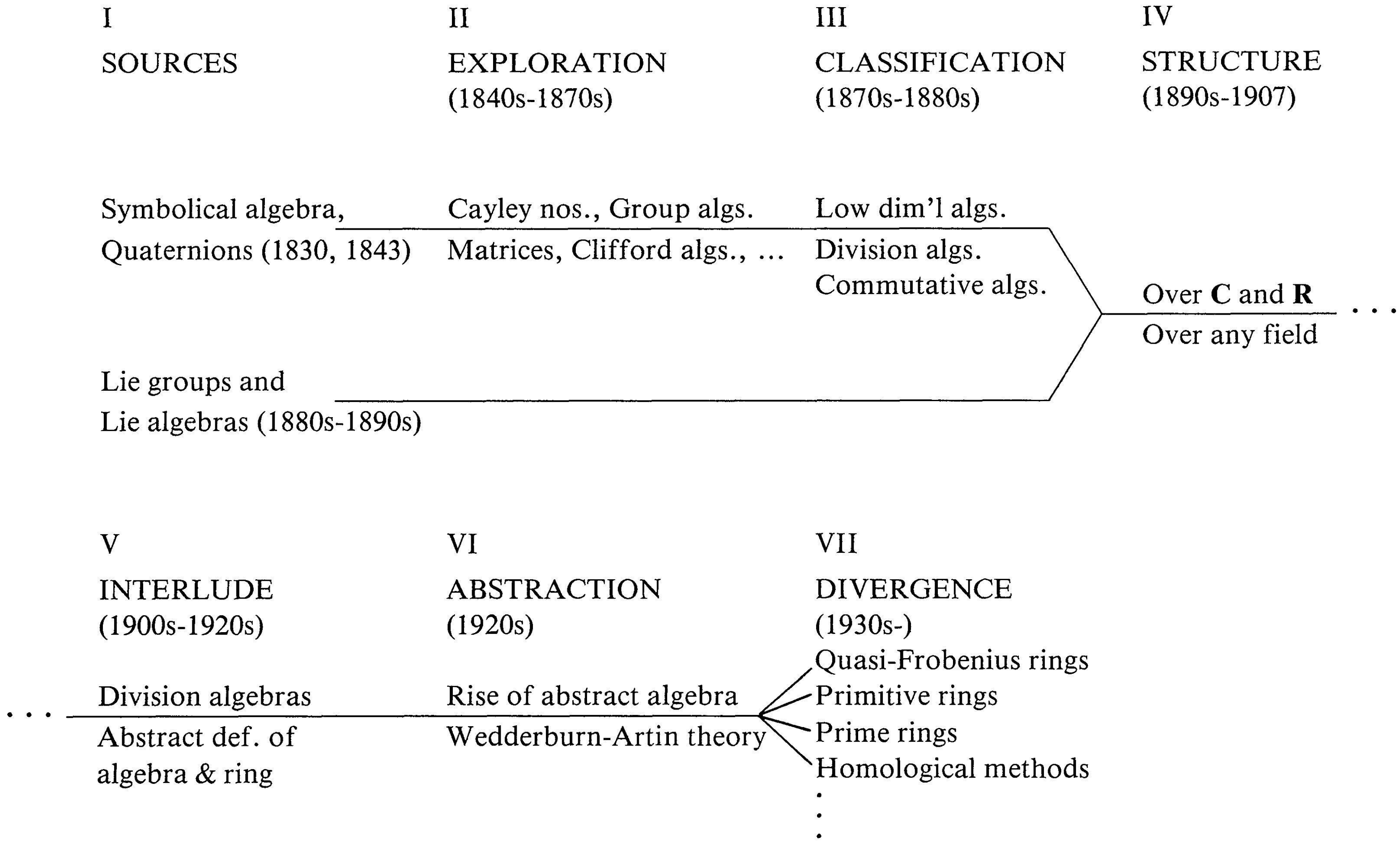
(g) Homological methods

Homological algebra is an offspring of algebraic topology, the latter, in turn, being inspired by the developments in abstract algebra and the suggestions initiated by Noether in the 1920s (see [57]). Among its fundamental concepts are the functors *Ext* and *Tor* which, in a sense, measure the manner in which modules over general rings "misbehave" when compared to the "nice" vector spaces of classical linear algebra. Other basic concepts of homological algebra are those of projective and injective module. (Every module is a homomorphic image of a projective module and a submodule of an injective module.) See [15] for details.

The debt that (algebraic) topology owed to algebra has been amply repaid. Homological (and category theory) methods have invaded much of abstract algebra, and especially ring theory—both commutative and noncommutative—beginning with the 1950s. It suffices to compare standard textbooks, let alone research monographs, of the 1950s (and even the 1960s) with those of the 1970s and 1980s (e.g. Jacobson's texts [45] and [48] to note the fundamental differences in language and technique. In fact, many of the standard concepts and results have been rephrased in homological language. For example, R is a division ring if and only if every R-module is free: R is semisimple and Artinian if and only if every R-module is injective, if and only if every R-module is projective, if and only if R has zero global (homological) dimension; R is quasi-Frobenius if and only if R satisfies the minimum condition and is injective as an R-module.

Finally, it might be of interest to mention that in addition to homological methods, analytic and topological methods have also invaded the study of rings; in fact, they have given rise to new branches of mathematics such as normed rings and differential rings (cf. Lie groups and topological groups). Another field related to that dealt with in this article is that of nonassociative rings; there are three important classes of such rings, giving rise to distinct theories, namely Lie rings, Jordan rings, and alternative rings. See e.g. [53].

We conclude this article with a diagrammatic sketch of the evolution of noncommutative ring theory as outlined in the various sections.


I
SOURCES
II
EXPLORATION
(1840s-1870s)
III
CLASSIFICATION
(1870s-1880s)
IV
STRUCTURE
(1890s-1907)
Symbolical algebra,
Quaternions (1830, 1843)
Cayley nos., Group algs.
Matrices, Clifford algs., ...
Low dim'l algs.
Division algs.
Commutative algs.
Over **C** and **R**
Over any field
Lie groups and
Lie algebras (1880s-1890s)
V
INTERLUDE
(1900s-1920s)
VI
ABSTRACTION
(1920s)
VII
DIVERGENCE
(1930s-)
Division algebras
Abstract def. of
algebra & ring
Rise of abstract algebra
Wedderburn-Artin theory
Quasi-Frobenius rings
Primitive rings
Prime rings
Homological methods