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Autor: Kleiner, Israel
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a ring such as $a \times 0 = 0$, and $a(-b) = (-a)b = -(ab)$. There are two extraneous axioms (dealing with "regular" elements in the ring) which depart from an otherwise modern definition.

Among the main concepts introduced are "zero divisors" and "regular elements". Fraenkel deals in this paper only with rings which are not integral domains and discusses divisibility for such rings. Much of the paper deals with decomposition of rings as direct products of "simple" rings (not the usual notion of simplicity).

Fraenkel's aim in this paper was to do for rings what Steinitz had just (1910) done for fields, namely to give an abstract and comprehensive theory of (commutative and noncommutative) rings.[1]) Of course he was not successful (he does admit that the task here is not as "easy" as in the case of fields)—it was too ambitious an undertaking to try to subsume the structure of both commutative and noncommutative rings under one theory. Fraenkel did, however, delineate the abstract notion of a ring and, in this respect, made a significant contribution.

VI. Structure of rings with minimum condition

In a fundamental paper of 1927 entitled "Zur Theorie der hyperkomplexen Zahlen" [5], Artin proved a structure theorem for rings with minimum condition (descending chain condition)[2]) which generalized Wedderburn's structure theorem for finite-dimensional algebras (discussed in sec. IV). The theorem, now known as the Wedderburn-Artin theorem for semi-simple rings with minimum condition (i.e. rings without nilpotent ideals and satisfying the descending chain condition for, say, right ideals—see e.g. [43]) states that if R is such a ring, then it is a direct sum of simple rings and these, in turn, are matrix rings over division rings; moreover, the above representations are unique (cf. Wedderburn's structure theorem, p. 246).

As we note, the *result* is essentially the same as Wedderburn's. It is, however, the spirit of the work and the conceptual advances which make it

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[1]) Steinitz' "Algebraische Theorie der Körper" of 1910 was the first abstract study of fields as a distinct subject. This fundamental work, which some say marked the beginning of modern abstract algebra, arose out of a desire to delineate the abstract notions common to the various contemporary theories of fields. It provided the basic concepts of field theory necessary for the subsequent abstract study of Galois theory, algebraic number theory, and algebraic geometry.

[2]) Artin proved his theorem for rings satisfying both the ascending and descending chain conditions. Later (1939) Hopkins showed that the descending chain condition suffices.

stand out as a very important contribution. Artin's work, though, must be seen against the background of the revolution in algebra which was taking place in the 1920s. It was initiated by Emmy Noether in a paper in 1921 on commutative rings with the ascending chain condition (now called Noetherian rings) entitled "Idealtheorie in Ringbereichen." We see here the beginnings of the conceptual, abstract, axiomatic approach to algebra. This is where the spirit (if not all the content) of so-called modern algebra originated.

Of course, E. Noether was not the first to use abstraction in algebra. Earlier (late 19th century) Dedekind had employed it in ideal theory and Galois theory; Frobenius, Weber *et al.* in group theory; and in the early 20th century, Wedderburn in associative algebra theory, and Steinitz in field theory. What Noether did was to bring unity and conceptual clarity to those developments. She highlighted what was essential in past work in abstract algebra by creating or bringing into prominence a number of central concepts and revealing hitherto unnoticed connections. To her abstract algebra was a distinct, conscious discipline, with its own concepts, methodology, and basic results. And, of course, the proof of her success was in the fertility of her approach, which animated not only algebra but also other branches of mathematics, such as topology, analysis, and number theory (eg. algebraic topology, functional analysis, and class-field theory, respectively). In Noether's own words [24]: "Algebra is the foundation and tool of all mathematics."

We briefly highlight three of her fundamental accomplishments. These still guide algebraic thinking today. Thus Noether

(a) Recognized the importance of *chain conditions*. In two great memoirs (in 1921 and 1927) on ideal theory, Noether founded the abstract study of rings with chain conditions.[1]) In the first (see above) she gave an abstract treatment of the decomposition theories of Hilbert, Lasker, and Macaulay for polynomial rings, and in the second (entitled "Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern") an axiomatic treatment of the theories of Dedekind and Kronecker for algebraic number and function fields. As Bourbaki noted [13]:

> It is seen in these memoirs how a small number of abstract ideas, such as the notion of irreducible ideal, the chain conditions, and the idea of an integrally closed domain ... can by themselves lead to general results

[1]) The ascending chain condition was introduced by Dedekind in connection with his study of ideals in an algebraic number field. Wedderburn, in his 1907 paper on the structure of algebras, uses "descending chain condition" arguments, without employing that term.

which seemed inextricably bound up with results of pure computation in the cases where they had previously been known.

(b) Gave prominence to the concept of *module*. Although this concept had been used earlier by Dedekind in concrete settings, Noether was the first to define it abstractly and note its importance as a unifying concept in algebra. In particular, she showed the usefulness of viewing representations as modules, which resulted in the absorption of the theory of group representations into the study of modules over rings and algebras (see [22]). Unlike earlier methods, this approach applied to fields of arbitrary characteristic. Noether also stressed that both ideal theory and the structure theory of algebras can be viewed as applications of module theory. This module-theoretic point of view, enabling the "linearization" of problems, has, of course, become fundamental.

(c) Highlighted the concept of *ring*. As we have noted, the concept of a ring was introduced in concrete settings by Dedekind and Hilbert and in the abstract by Fraenkel. It was Noether, however, who, through her groundbreaking papers, in which the concept of ring played a fundamental role, brought this concept into prominence as a central concept of algebra, taking its rightful place alongside those of group and field. The concept of ring immediately began to serve as the starting point for much of the development of abstract algebra that followed.

Let us conclude our account of Noether's work with several testimonials:

> Emmy Noether was one of the most influential mathematicians of this century. ...The development of abstract algebra, which is one of the most distinctive innovations of 20th century mathematics, is largely due to her. (N. Jacobson, in the introduction to E. Noether's collected works.)
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> She taught us to think in simple and thus general terms ... homomorphic image, the group or ring with operators, the ideal ... and not in complicated algebraic calculations; and she therefore opened up a path to the discovery of algebraic regularities where before these regularities had been obscured by complicated specific conditions. (P. Alexandroff [15].)
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> The methodological concepts of arithmetization, generalization, abstraction, reduction, and transfer are the spindles she used to trim and combine in an orderly fashion the algebraic threads that had been generated, separated, and entangled with geometric and analytic strands during the preceding century. (U. Merzbach [75].)

Artin was himself a major contributor to the concepts, methods, and results of abstract algebra during the crucial decade of the 1920s. His structure theorem for rings with the descending chain condition is, in fact, a model in the spirit of this period. The features to note are:

(a) The *ring* rather than the algebra becomes the central object of study. Noether began to prepare the ground (with her 1921 paper) for the ascendancy of the concept of ring. Her work, together with Wedderburn's conceptual treatment of his structure theorems for algebras, made it "natural" (in the hands of a master like Artin) to extend the theorems to rings (with minimum condition). The concept of ring now becomes central.

(b) *Chain conditions* acquire prominence. In Noether's papers of 1921 and 1927 the *ascending* chain conditions for ideals is the central notion. In Artin's work the *descending* chain condition is introduced for the first time and acquires importance.

(c) *One-sided ideals* are used as an essential tool. Artin's rings, contrary to Noether's, are noncommutative. Here the concept of one-sided ideal, briefly introduced in Wedderburn's 1907 paper (as "semi-invariant subalgebra"), acquires central importance. In fact, the chain conditions in Artin's paper apply to one-sided ideals.

Artin's theorem proved to be a model and an essential tool in subsequent work on the structure of rings. In the words of Herstein [34]:

> [The Wedderburn-Artin structure theorem] is the cornerstone of many things done in algebra. From it comes out the whole theory of group representations. In fact there are very few places in algebra—at least where noncommutative rings are used—where it fails to make its presence felt.

The new ideas in abstract algebra of Artin, Noether, and others were disseminated in 1930-31 in an influential book by Van der Waerden entitled "Modern Algebra" [77]. G. Birkhoff [10] gives an absorbing account of its impact on the wider mathematical community:

> Even in 1929, its concepts and methods [i.e. of "modern" algebra] were still considered to have marginal interest as compared with those of analysis in most universities, including Harvard. By exhibiting their mathematical and philosophical unity and by showing their power as developed by Emmy Noether and her other younger colleagues (most notably E. Artin, R. Brauer, and H. Hasse), van der Waerden made "modern algebra" suddenly seem central in mathematics. It is not too much to say that the freshness and enthusiasm of his exposition electrified the mathematical world—especially mathematicians under 30 like myself.