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of a number of copies of either **R** or **C**. Thus one not only adds but also multiplies the elements of the algebra componentwise (when they are given as $\sum a_i e_i$). An immediate consequence of this result is that the only commutative division algebras over **R** are **R** or **C**. (This latter result also follows, of course, from that of Frobenius/Peirce in (b) above.)

The above characterization of commutative algebras over **R** and **C** was obtained, independently, by Weierstrass and Dedekind in the 1860s, although their works were published only in 1884-85 (see [80]). Both men were motivated, at least in part, by the following remark of Gauss, made in an 1832 paper on complex numbers [66]:

> The author [Gauss] has reserved for himself [the task] of working out more completely the subject, which in the present treatise is actually only occasionally touched upon. There then, too, the [following] question will find its answer: Why can the relations between things which present a multiplicity of more than two dimensions not furnish still other kinds of quantities permissible in the general arithmetic?

It is remarkable that Gauss seems to have anticipated here (as also, of course, in connection with fundamental developments in other branches of mathematics) the study of hypercomplex systems, and the fact that there are no systems analogous to **C** (i.e. fields) whose dimensions are greater than 2.

Dedekind's interest in commutative rather than general hypercomplex systems is understandable. In his fundamental work on ideal theory in algebraic number fields, Dedekind views the number field as an extension of the field of rational numbers, hence as a finite-dimensional algebra over the rationals. He exploits this point of view in his studies of algebraic number theory. To Dedekind, then, a finite-dimensional commutative algebra was a familiar object.

Dedekind's work helped to stimulate the deeper works of Molien and Cartan on the structure of more general types of algebras. This is part of the story to which we turn next.

IV. STRUCTURE OF ALGEBRAS

The first example of a noncommutative algebra was given by Hamilton in 1843. During the next forty years mathematicians introduced other examples of noncommutative algebras, began to bring some order into them and to single out certain types of algebras for special attention. Thus low-dimensional algebras, division algebras, and commutative algebras were classified and characterized. The stage was (almost) set for the founding of a general theory of finite-dimensional, noncommutative, associative algebras. The task was accomplished in the last decade of the 19th century and the first decade of the 20th century. Before that, however, important developments took place in a neighboring branch of mathematics which had an impact on the work in associative algebras. This was the founding of the theory of Lie groups and Lie algebras in the 1870s and 1880s, to which we shall shortly turn (see source (b) in the outline on p. 228 and the remark following on pp. 229-230). But first a description of the structure theory of associative algebras—the main subject of this section IV.

If A is a finite-dimensional associative algebra, we have the following result:

(a) $A = N \oplus B$, where N is nilpotent and B is semi-simple.[1]) (An algebra N is *nilpotent* if $N^k = 0$ for some positive integer k. An algebra is *semi-simple* if it has no nontrivial nilpotent ideals—this, at least, was the initial conception of semi-simplicity.)

(b) $B = C_1 \oplus C_2 + \ldots \oplus C_n$, where C_i are simple algebras (i.e. have no nontrivial ideals).[2])

(c) $C_i = M_{n_i}(D_i)$, the algebra of $n_i \times n_i$ matrices with entries from a division algebra D_i.

The above representations are, moreover, unique (i.e. the n, n_i are unique, and the N, B, C_i, D_i are unique up to isomorphism.)

These results were derived for algebras over **R** and **C** by Molien, Cartan, and Frobenius, and about ten years later for algebras over an arbitrary field by Wedderburn. It should be noted that this type of structure theorem (i.e. the decomposition of a structure into "simple" substructures) was not new to algebra. Although the immediate inspiration and motivation for this result came from the neighboring theory of Lie algebras, there were other precedents.[3]) In the Lie algebra case, Killing and Cartan made the major

[1]) The decomposition of A into N and B does not hold for algebras over arbitrary fields although it holds over **R** & **C**. See [3] for the conditions under which it is true.

[2]) The nilpotent part N is intractable, even today.

[3]) The decomposition of an integer into a unique product of primes is the first such instance and goes back to Greek antiquity. In the 1870s Dedekind gave a decomposition of ideals in an algebraic number field into prime ideals. The decomposition of a finite abelian group into a direct product of cyclic groups of prime power order, given by Frobenius and Stickelberger in 1879, is another example.

contributions in the 1880s by decomposing the semi-simple Lie algebras into simple ones and then classifying the simple Lie algebras (see [18], [47]). Cartan was thus a major contributor to both the Lie and associative theories, and, in addition to the model, he also carried over some of the techniques from the Lie to the associative case. There were also *direct* connections between Lie *groups* and hypercomplex number systems, to which we turn shortly. But first a word about the origin of Lie groups and Lie algebras.

Lie founded the theory of continuous transformation groups (what we today call Lie groups) in the 1870s so as to facilitate the study of differential equations (cf. Galois theory). Just as Galois associated a finite (discrete) group of permutations with an algebraic (polynomial) equation, so Lie associated an infinite (continuous) group of transformations with a differential equation. Lie subsequently showed that for the purposes of the differential equation it suffices to focus on the "local" structure of the Lie group—that is, on the "infinitesimal transformations" which, when multiplied using the "Lie product", form a Lie algebra. (If S, T are infinitesimal transformations, so is their Lie product $[S, T]$ which is given by $[S, T] = ST - TS$.) Just as in the case of algebraic equations, so too in this theory the objects of special interest are the "simple" Lie groups. These give rise to "simple" Lie algebras. Lie thus proposed the task of studying the structure of Lie algebras with special attention to be given to simple Lie algebras. This task (as we mentioned) was admirably accomplished by Killing and Cartan.

In an influential two-page paper of 1882 called "Sur les nombres complexes" Poincaré highlighted the connection between Lie groups and hypercomplex systems:

> The remarkable works of M. Sylvester on matrices have recently drawn attention to complex numbers [hypercomplex number systems] analogous to Hamilton's quaternions. The problem of complex numbers is easily reduced to the following: to find all the continuous groups of linear substitutions in n variables, the coefficients of which are linear functions of n arbitrary parameters.

Poincaré had in mind the following: If H is a hypercomplex system, then every element $u \in H$ determines a linear transformation $u_R: H \to H$ given by $u_R(x) = xu$. Poincaré then wanted to determine all possible continuous groups of such transformations; in modern terms, he was interested in finding all subgroups of the Lie group $GL_n(\mathbf{C})$ of invertible $n \times n$ matrices over $\mathbf{C}$. (Note that u_R, if invertible, can be thought of as an element of $GL_n(\mathbf{C})$.) Thus a hypercomplex system gives rise to a Lie group. Conversely, given an

(n-parameter) Lie group $G = \{T_u : u = (u_1, \ldots, u_n)$ is a parameter$\}$ (the elements of a Lie group were given as continuous transformations

$$x_i' = f_i(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_n)$$

of n variables x_i, and n parameters $u_i, i = 1, 2, \ldots, n$), the multiplication $T_u T_v = T_w$ in G determines a product $uv = w$ of the parameters (n-tuples). The question is: when does this represent multiplication in a hypercomplex number system? [1]) See [18], [27], [40], [47], [80] for details.

Of the three mathematicians—Molien, Cartan, Frobenius—who gave the structure theorems for associative algebras over the real and complex fields, Cartan was the most influential, so we shall focus (briefly) on his contributions. For details see [40], [66], [80].

Cartan's work on the subject appeared in an 1898 paper entitled "Sur les groupes bilinéaires et les systèmes de nombres complexes". As the title indicates, the paper deals with the relation between hypercomplex systems and Lie groups (see the comments above on this relationship). The thrust and major part of the paper, however, are directed to a development of the structure of hypercomplex systems *independent of Lie group theory*. Both Scheffers and Molien had previously obtained many of the results which now appeared in Cartan's work on the structure of such systems. (Cartan was apparently unaware of Molien's work. For details of both Scheffers' and Molien's contributions see [66].) Scheffers and Molien, however, had used techniques and results from the Lie theory to study the associative case. It was Cartan's expressed aim to avoid this. As he says, his development of the structure theory of associative algebras "does not call forth any notion from the theory of [Lie] groups and stays exclusively in the realm of the theory of [hyper]complex numbers." Cartan's main contribution, then, lies in the development of methods and concepts *internal* to the theory of hypercomplex number systems.

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[1]) The association of one type of mathematical structure with another so as to study the latter by means of the former is, of course, a very fruitful idea in mathematics. We have already noted the association of Lie groups to hypercomplex systems, Lie groups to Lie algebras, differential equations to Lie groups, algebraic equations to finite groups. Other examples are the associations between equations and curves (analytic geometry), groups and fields (Galois theory), groups and geometries (Klein's Erlangen Program), topological spaces and homology groups (algebraic topology), groups and matrices (representation theory). Moreover, any associative algebra can be turned into a Lie algebra by defining in it a new product, the so-called Lie (or bracket) product $[x, y] = xy - yx$. Conversely, any Lie algebra can be embedded in an associative algebra, the so-called Birkhoff-Witt enveloping algebra of a Lie algebra, in which the Lie product corresponds to the product in the Lie algebra. We thus have an association between Lie algebras and associative algebras.

Cartan associates a characteristic and minimal polynomial with each *algebra* (see e.g. [3])—these are fundamental tools in his development of the theory. Their factors are related to the structure of the given algebra.[1]) Thus the underlying vector space of the associative algebra, and hence the fields of scalars **R** and **C**, play a fundamental role in Cartan's work. Not only his proofs but also some of his concepts rely on the linear algebra structure. For example, Cartan defines a "pseudo-null" element of an algebra as one whose characteristic polynomial has only the zero root. It can be shown that this notion is equivalent to that of a nilpotent element (see [66], p. 297), defined almost thirty years earlier by Benjamin Peirce, although Cartan does not make that identification. Cartan's proofs, then, are not conceptual and are often quite long (e.g. some five-page proofs of Cartan can be replaced by five-line proofs in the style of Wedderburn, whose work we shall describe shortly). What proved lasting in Cartan's work in addition to the main structure theorem, though he apparently did not attach to them great significance, were the four concepts which he introduced at the end of this development in order to *state* his structure theorem more succinctly. These were the notions of direct sum, two-sided ideal,[2]) and simple and semi-simple algebra. (Others before Cartan used these concepts, but only implicitly.) For example, Cartan defines an ideal (he calls it an "invariant subsystem") as follows:

> We say that a system Σ admits an invariant subsystem σ, if every element of σ belongs to Σ and if the product, on the right or on the left, of an arbitrary element of σ and an arbitrary element of Σ belongs to σ.

Cartan's description of simple algebras (which he defined as those having no invariant subsystems) is as follows:

> All simple systems of complex numbers are of the same kind; they are formed by p^2 basis elements e_{ij}, where i and j take on all values $1, 2, \ldots, p$ and the law of multiplication of these basis elements is given by the formulas $e_{ij}e_{jl} = e_{il}$, $e_{ij}e_{\lambda l} = 0 (j \neq \lambda)$. The number p is an

[1]) Recall the well-known result from linear algebra that if V is a finite-dimensional vector space and $T: V \to V$ a linear transformation with characteristic polynomial $f(t) = (t-\lambda_1)^{m_1} \ldots (t-\lambda_r)^{m_r}$, then $V = V_1 \oplus \ldots \oplus V_r$, where $V_i = \{x \in V: (T-\lambda_i I)^n x = 0$ for some $n\}$. This gives an indication of the kind of ideas involved.

[2]) Dedekind introduced the notion of an ideal for rings of algebraic integers in the 1870s, in connection with his fundamental work in algebraic number theory. Kronecker dealt with ideals (which he called "divisors" and "modular systems") of polynomial rings in 1882. Both of these types of rings are, of course, commutative. There is, in any case, no mention in Cartan's work of the ideals of Dedekind or Kronecker.

arbitrary integer greater than or equal to 1. These systems are called p^2-ions.[1])

Wedderburn's work (1907)

At the end of the 19th century the theory of hypercomplex number systems had attained a degree of maturity. All-important connections had been made with Lie's theory of continuous groups (as we have seen), as well as with the theory of finite groups, via group representation theory.[2]) At the same time a major structure theorem was available, established without recourse to external theories. Partly because of this, and partly because of the connections with other major fields, the theory of hypercomplex systems became a distinct discipline for serious mathematical investigation (on both sides of the Atlantic). What was needed now was a new departure. This was provided by Wedderburn's groundbreaking paper of 1907 entitled "On hypercomplex numbers" [82].

It has been said that a "good abstract theory" is one which summarizes and unifies previous results, placing them in a new perspective, and one which provides new directions for subsequent work in the field. Without doubt, Wedderburn's work qualifies as such.[3])

The major result in Wedderburn's paper, namely the structure theorem for finite-dimensional associative algebras, is essentially the same as that given by Cartan (see p. 242). There was "merely" an extension of the field of scalars of the algebra from **R** and **C** to an arbitrary field. This extension, however, necessitated a new approach to the subject—a rethinking and reformulation of the major concepts and results of the theory of hypercomplex number systems.

Wedderburn came to the University of Chicago from Scotland in 1904 as a Carnegie Research Fellow. Here he met E.H. Moore, Bolza, Maschke, and Dickson, all of whom provided great inspiration for the 22-year-old Wedderburn. This very strong school of algebra at the University of Chicago excelled

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[1]) It is interesting that the language of matrices, introduced in the 1850s, was not very familiar even as late as 1898.

[2]) A basic tool in the study of group representations is the group algebra of a finite group. This is a semi-simple hypercomplex system (at least over **R** and **C**) to which the structure theorem of Cartan can therefore be applied. These ideas were fundamental in the development of group representation theory. See [40] for details.

[3]) Other outstanding examples of "good abstract theories" in algebra are Steinitz' theory of fields of the 1910s (see footnote on p. 253), E. Noether's theory of commutative rings with the ascending chain condition of the 1920s, and Artin's formulation of Galois theory in the 1930s.

in the abstract Anglo-American tradition, following in the footsteps of such luminaries as Boole, Cayley, B. Peirce, and C.S. Peirce (see [8], [65]). This was, indeed, fertile ground for the young Scotsman.

Wedderburn clearly recognized that his major contributions lay in the *methods* he was developing to prove the structure theorems. In his own words:

> The object of this paper is in the first place to set the theory of hypercomplex numbers on a rational basis.[1]) The methods usually employed in treating the parts of the subject here taken up are, as a rule, dependent on the theory of the characteristic equation and for this reason often valid only for a particular field or class of fields. Such, for instance, are the methods used by Cartan...

We now give a brief summary of some of Wedderburn's main concepts, methods, and results.

(1) The notion of *ideal* is central to the study. Although, as we noted, the concept of ideal appears in Cartan's work (and, to some extent, also in Molien's and Frobenius'), it is only with Wedderburn that it is given a central place in the study of algebras. (Recall that Cartan defines ideals towards the *end* of his paper so that he can *state* his results succinctly.) "The theory of invariant sub-algebras [ideals]", Wedderburn says, "is of great importance." In fact, it forms the essence of the "rational basis" upon which Wedderburn founds his work, replacing the "auxiliary" characteristic and minimal polynomials of Cartan.

(2) The concept of *difference (quotient) algebra* is defined. Wedderburn mentions the analogy with the quotient group of a finite group, which was introduced in 1889 (see [49]).[2]) Among other results, he shows that if B is a maximal invariant subalgebra of A, then A/B is a simple algebra.

(3) The concept of *nilpotent algebra* is introduced. "Nilpotent algebras are of great importance in the discussion of the structure of algebras", Wedderburn notes. (Recall Cartan's "pseudo-null" [nilpotent] elements which he defined in terms of the roots of the characteristic polynomial.) Wedderburn

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[1]) That is, independent of the field of scalars. In fact, toward the end of the paper Wedderburn says that "It is remarkable that the properties of a field with regard to division are not used in many of the theorems of the preceding sections".

[2]) The notion of quotient structure is implicit in Gauss' congruences modulo n of 1801. This idea was modelled by Cauchy in 1847 in his definition of the complex numbers as congruence classes of real polynomials modulo $x^2 + 1$. Kronecker, in connection with his work in algebraic number theory and algebraic geometry in the 1880s, generalized Cauchy's device to quotient rings of polynomial rings (in any number of indeterminates) with respect to their ideals.

shows that every algebra contains a maximal nilpotent invariant subalgebra (it may, of course, be zero) which contains all other nilpotent invariant subalgebras. This is what we today call the *radical* of the algebra (if the algebra if finite-dimensional), although Wedderburn does not have a special name for it. (The term "radical" is due to Frobenius—it appears in his 1903 work on the structure of hypercomplex number systems. The *notion* of radical was adumbrated earlier by Molien and Cartan.)

(4) A *semi-simple algebra* is defined as today. Cartan and others *defined* a semi-simple algebra as a direct sum of simple algebras. For Wedderburn, "Algebras which have no nilpotent invariant sub-algebra form a very important class. Such algebras are called *semi-simple.*"

Wedderburn shows that if one "factors out" the radical of an algebra one gets a semi-simple algebra. "This theorem", he claims, "is very important, its importance lying in the fact that... it enables us to confine our attention to algebras which have no nilpotent invariant subalgebra." This technique was central to his successful study of the structure of algebras. By factoring out the radical and focusing on the well-behaved semi-simple part of the algebra, Wedderburn was able to arrive efficiently and conceptually at his main results, "whereas CARTAN microscopically dissected his algebra and found himself entangled in the extremely complicated structure of the radical" (Parshall [66]).

Among the results which Wedderburn establishes for a semi-simple algebra A are the following:

(i) A is a direct sum of simple algebras.

(ii) If e is an idempotent element of A then eAe is semi-simple.

(iii) A contains an identity element.

(5) Explicit recognition is given to the concept of a *division algebra.* Since in the past algebras were considered only over **R** or **C**, the concept of a division algebra was of no special consequence. (As we have noted, there is only one (finite-dimensional) noncommutative division algebra over **R**, namely the quaternions, and there are none over **C**—as Wedderburn shows in this paper; see sec. V below). Wedderburn's structure theorems for algebras over an arbitrary field highlight, however, the concept of a division algebra. Moreover, as Wedderburn says, "[The structure theorems are] incomplete in so far as the classification is given in terms of primitive [division] algebras which have themselves not yet been classified."

Wedderburn defines a division algebra as one having a single idempotent element and no nilpotent elements, and then shows that in such an algebra every nonzero element is invertible. He also shows that if e is a primitive idempotent of an algebra A (he defines e to be "primitive" if it is the only idempotent in A—this is equivalent to the modern definition; see [3]), then eAe is a division algebra.

(6) The *tensor product* of algebras is introduced. Although Scheffers made use of this notion in 1891, Wedderburn was the first to define formally and explicitly the important notion of the tensor product of two algebras:

> If C and D are any algebras such that every element of the one is commutative with every element of the other, and if the order of the complex [subspace] $A = CD$ is the product of the orders of C and D, then A is an algebra which is called the *direct* [tensor] *product* of C and D.

Wedderburn then states his main theorem on the classification of simple algebras in terms of the tensor product:

> Any simple algebra can be expressed as the direct [tensor] product of a primitive [division] algebra and a simple matric algebra.

Wedderburn also proves the converse of this theorem, namely:

> The direct product A of a primitive algebra B and a quadrate matric algebra C is simple.

Wedderburn called the algebra of $n \times n$ matrices over a field a "simple or quadrate matric algebra of order n^2."

(7) The *Peirce decomposition* of an algebra is extended. For Wedderburn, just as for his predecessor Peirce, idempotents play a central role in the study of algebras. Wedderburn extends the Peirce decomposition of an algebra (see p. 239) by the use of (what we call today) pairwise orthogonal primitive idempotents (see [3], [46]), and is thus able to minutely dissect the algebra.

Eighty years after Wedderburn's work his concepts, methods, and results are still basic to our study of algebras (but see sec. VII (g) concerning homological methods). Moreover, Wedderburn's paper can easily be read by today's student. (By contrast, reading, say, Cartan's work—only ten years earlier—on the subject is a formidable task.)

Wedderburn provided, indeed, a truly "rational basis" for the study of the structure of algebras. By bringing into relief the important concepts and methods of the theory, his work invited the generalization by Artin to rings with minimum condition (see sec. VI below). What it left open is important questions about the structure of nilpotent algebras and division algebras (see secs. V, VII). It is perhaps fitting to conclude our discussion of Wedderburn's work with several tributes:

> [Wedderburn] was the first to find the real significance and meaning of the structure of a simple algebra. This extraordinary result has excited the fantasy of every algebraist and still does so in our day. (Artin, [6].)

> Wedderburn's pioneering work on the structure of simple algebras set the stage for the deep investigations—often with an eye to applications in algebraic number theory—in the theory of algebras. (Herstein, [43].)

> [Wedderburn's] rational methods struck at the heart of the theory of algebras, and their influence is felt even to this day... His work neatly and brilliantly placed the theory of algebras in the proper, or at least in the modern, perspective. (Parshall, [66].)

V. Interlude

The title is not meant to suggest a lack of activity in the study of algebras during the first two decades or so of the 20th century. There were simply no *fundamental* developments in the period between the work of Wedderburn in 1907 and the works of Artin, Noether *et al.* in the 1920s. Below we briefly describe two areas of progress in these intervening years.

(a) Division algebras

As we noted, Wedderburn's structure theorems left unresolved the nature of division algebras. Knowledge of finite-dimensional division algebras over a field F was available only in the following three cases:

(i) $F = \mathbf{R}$. In this case, as we have seen, there are only three division algebras over F, namely the reals, complex numbers, and quaternions.

(ii) $F =$ an algebraically closed field (eg. $\mathbf{C}$). In this case Wedderburn himself showed (in the 1907 paper where his structure theorem appears) that over such a field there are no division algebras except for the field itself. As Wedderburn put it:

> If the given field is so extended that every equation is soluble, the only primitive [division] algebra in the extended field is the algebra of one unit, $e = e^2$.