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Artikel: SKETCH OF THE EVOLUTION OF (NONCOMMUTATIVE) RING THEORY

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Kapitel: I. Sources

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IV. *Structure of algebras*

(a) Over the complex and real numbers (Molien, E. Cartan, Frobenius); 1890s-1903.

(b) Over any field (Wedderburn); 1907.

V. *Interlude*

(a) Division algebras (Wedderburn, Dickson); 1905-1926.

(b) Definitions of abstract algebras and abstract rings (Dickson, Fraenkel); 1903, 1914.

VI. *Structure of rings with minimum condition*

The Artin-Wedderburn structure theorems (Artin); 1927.

VII. *Some subsequent developments*

(a) Deep study of division rings (algebras) (Albert, Brauer, Hasse, Noether); late 1920s-early 1930s.

(b) Nilpotent rings; 1930s-.

(c) Quasi-Frobenius rings (Nakayama); 1939-1941.

(d) Primitive rings (Jacobson); 1945.

(e) Prime rings (Goldie); 1958-1960.

(f) Representations of rings and algebras (Köthe, Brauer *et al.*); 1930s-.

(g) Homological methods (H. Cartan, Eilenberg, MacLane *et al.*); 1950s-.

We now turn to a point by point discussion of the above outline.

I. Sources

The early sources—symbolical algebra and quaternions—provided the impetus for the birth and early growth of the theory of hypercomplex number systems, while the later sources—Lie groups and Lie algebras—supplied the inspiration, motivation, and techniques for its mature development, resulting in the basic structure theorems for such systems (beginning with IV above). These two sources also reflected two distinct approaches to the subject of hypercomplex systems, two traditions—the abstract Anglo-American tradition and the more "down-to-earth" concrete Continental tradition, respectively (see [66]). We shall deal with the later sources (Lie groups and Lie algebras) when they begin to make their impact (under item IV).

In a strict sense, noncommutative ring theory originated from a single example, namely the quaternions created by Hamilton in 1843. These are "numbers" of the form $a + bi + cj + dk$ (a, b, c, d real numbers) which are added componentwise and multiplied according to the rules $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, with the obvious extension to all quaternions. This was the first example of a noncommutative "number system" (a system closed under the operations of addition, subtraction, multiplication and, initially, also division), and was the catalyst for the introduction of various other noncommutative number systems. We will consider some of these, as well as Hamilton's motivation for his creation, shortly. We wish to turn briefly now to an earlier development, namely the creation of symbolical algebra in the 1830s, which brought about, through speculations on the nature of algebra, a favourable climate for the construction of various "nontraditional" number systems.

The isolation of British mathematics in the 18th century had adverse effects on its development. The universities, for example, taught Newton's method of fluxions rather than the more powerful methods developed on the continent by the Bernoullis, Euler, and Lagrange. In the early 19th century several British mathematicians at Cambridge, among them Whewell, Peacock, and De Morgan, undertook a reform in the teaching of mathematics. In the British universities of this period mathematics was viewed more as an instrument for the training of logical minds than as a tool for the solution of practical problems. If this was to be effective, one would have to put algebra (which at the time consisted of symbolical manipulation relating to numbers) on a sound logical basis. That meant *justifying* the laws of operation with numbers (mainly negative and complex numbers). For example, why is $(-a)(-b) = ab$? Is $\sqrt{ab} = \sqrt{a}\sqrt{b}$ true for negative as well as positive numbers? Why is $a(b-c) = ab - ac$ when $b < c$? Why is $a^m a^n = a^{m+n}$ when m and n are negative or rational numbers?

The applicability of the "laws" of arithmetic to positive integers was considered "obvious". (For example, the equality $a(b-c) = ab - ac$ was not questioned when $b > c$.) The "science" of operations on the symbols representing positive integers was called "arithmetical algebra", and, as mentioned, was taken for granted. Against this, Peacock, De Morgan, Gregory, and others created "symbolical algebra", which was intended to have much broader applicability. We focus on Peacock's work, which had the greatest impact. It was embodied in his major publication *A Treatise on Algebra* of 1830.

Peacock wrote this work, he says, "with a view of conferring upon Algebra the character of a demonstrative science". To him symbolical algebra was "the science which treats of the combination of arbitrary signs and symbols by means of defined though arbitrary laws". As for the laws, he claims that

> We may assume any laws for the combination and incorporation of such symbols, so long as our assumptions are independent, and therefore not inconsistent with each other.

This, of course, is a rather modern point of view, well ahead of its time. This programme, however, was not implemented by Peacock in the stated generality. For although Peacock seems to profess freedom in choosing the laws which such a symbolical algebra can possess, he in fact postulates these laws to be the same as the laws of arithmetic. Thus, for example, the equality $a(b-c) = ab - ac$ is *decreed* to be true when $b < c$. We will not go into details here since the subject has been dealt with extensively in the literature. See [10], [15], [51], [60], [61], [68], [70], [71], [73].

To summarize, the doctrine of the British School of symbolical algebra, although not fully implemented, suggested that what matters in algebra are the rules which symbols obey rather than the meaning which one may attach to such symbols.[1]) Whatever its limitations, it provided a positive climate for subsequent developments in algebra. Thus symbolical algebra can be said to have given mathematicians the licence to create nontraditional number systems, while Hamilton and others showed them how to do it. In the words of Bourbaki [13]:

> The algebraists of the English school were the first to isolate, between 1830 and 1850, the abstract notion of law of composition, and then immediately broadened the field of algebra by applying this notion to a host of new mathematical entities: the algebra of logic (Boole), ... quaternions (Hamilton), matrices and nonassociative algebras (Cayley).

Hamilton, in his creation of quaternions in 1843, would seem to have practiced what Peacock had preached. This is not the case, however. Hamilton was, indeed, concerned with the foundations of algebra. As he remarked in 1853 (see [35]):

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1) This notion, that symbols have a "life of their own", without requiring recourse to meaning, has its precursor in the Operational Calculus of Lagrange and the work of others in the 18th century. It has been suggested that the inspiration for symbolical algebra came from these sources. See [51].

> The difficulties which so many have felt in the doctrine of Negative and Imaginary Quantities in Algebra forced themselves long ago on my attention.

Hamilton's proposed solution of these difficulties, however, was different from that of Peacock and his school of symbolical algebra. In particular, Hamilton objected to Peacock's view of the symbols of algebra as arbitrary marks without any *meaning*. To Hamilton the symbols of algebra had to stand for something "real"—not necessarily material objects but at least mental constructs. It was necessary, Hamilton claimed, to "look beyond the signs to the things signified." [1]) (Isn't this a foreshadowing of the formalist—intuitionist debate of almost a century later?)

Hamilton formulated his "philosophy" of algebra in an 1835 paper entitled "Theory of Conjugate Functions, or Algebraic Couples; With a Preliminary and Elementary Essay on Algebra as the Science of Pure Time." He was stimulated to write this essay, he says, by passages in Kant's *Critique of Pure Reason*. Briefly, just as (according to Kant) geometry is grounded in a mental intuition of space, so algebra should be grounded in a mental intuition of time. (For details see [35], [63], [68]). In this essay Hamilton also defines the complex numbers as ordered pairs of real numbers. Here he presents one of the earliest attempts to list systematically the properties of the real and complex number systems. He comes very close, in fact, to defining a field—the commutative and distributive laws are given, as are the closure laws, additive and mutiplicative inverses, and a definition of the zero element. What is missing are the associative laws—these he stated for the first time in his 1843 work on quaternions, to which we now turn.

Hamilton's motivation for the introduction of the quaternions is not easy to reconstruct. (For the *mathematical* thinking leading to their creation see [79].) His own (retrospective) view of them, given in 1855, is (see [35]):

> The quaternion [was] born, as a curious offspring of a quaternion of parents, say of geometry, algebra, metaphysics, and poetry... I have never been able to give a clearer statement of their nature and their aim than I have done in two lines of a sonnet addressed to Sir John Herschel:
>
> > "An how the one of Time, of Space the Three. Might in the Chain of Symbols girdled be."

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[1]) It is interesting that the mathematicians of the symbolical school soon came to think that Hamilton's work was well in accord with their own philosophy of algebra.

We might surmise that the geometric motivation was the desire to extend vectors in the plane to vectors in space (an extension which the quaternions, in a sense, accomplished); the algebra stemmed from a natural (from a mathematician's point of view) desire to extend number-pairs to triples, and, when this failed, to quadruples; the metaphysical connection with the ideas of Kant, which we mentioned above, was a factor in all of Hamilton's works in algebra; as for the poetry, we can do no better than to quote Weierstrass: "No mathematician can be a complete mathematician unless he is also something of a poet". See [35] for a detailed analysis.

For twenty two years following the invention of the quaternions, Hamilton was preoccupied almost exclusively with their application to geometry, physics, and elsewhere. To him the quaternions were the long-sought key which would unlock the mysteries of geometry and mathematical physics. The main importance of the quaternions, however, lay in another direction, namely in algebra (see [56]). Poincaré's tribute of 1902 is telling:

> Hamilton's quaternions give us an example of an operation which presents an almost perfect analogy with multiplication, which may be called multiplication, and yet it is not commutative... This presents a revolution in arithmetic which is entirely similar to the one which Lobachevsky effected in geometry.

We will explore some of the consequences of that revolution in the following section.

II. Exploration

Hamilton's quaternions at first received less than universal understanding and acclaim. Thus when Hamilton communicated his invention (discovery?) to his friend John Graves, the latter responded as follows [66]:

> There is still something in this system [of quaternions] which gravels me. I have not yet any clear view as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties.

Most mathematicians, however (including Graves) quickly came around to Hamilton's point of view. The floodgates were opened and the stage was set for the exploration of diverse "number systems", with properties which