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A SKETCH OF THE EVOLUTION OF (NONCOMMUTATIVE) RING THEORY

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The evolution of noncommutative ring theory [1]) spans a period of about one hundred years, beginning in the first half of the 19th century. This period also saw the evolution of the other major algebraic theories, namely group theory, commutative ring theory, field theory, and linear algebra. The evolution of noncommutative ring theory must be seen within this broader context of the evolution of "abstract algebra" as a whole. Results and concepts from one area were often carried over to another. We will indicate some of these interconnections throughout the paper. More generally, the emergence of a critical climate in mathematics in the 19th century no doubt had an impact on the developments which we shall relate.

The high point in the "classical" theory of (noncommutative) rings is the Wedderburn-Artin structure theorem. It gives the structure of an important class of rings in terms of "simple" rings. Wedderburn's contribution to this theorem is detailed in section IV of this article (see the outline below), Artin's in section VI. The theory of rings can be said to have reached its *maturity* in this 20-year period (1907-1927). The first three sections of the article deal with the *birth* and *growth* of the theory of rings (ca. 1830-the end of the 19th century), while section VII gives an indication of some of the "modern" (post 1930) aspects of the theory.

We will also note the mathematical backdrop against which all these developments took place. Thus, the initial growth of the theory occurred in the positive climate following the achievements of the British school of symbolical algebra, the work of Cartan at the end of the 19th century was engendered by important developments in the related area of Lie groups and Lie algebras, Wedderburn was inspired by the strong algebra school at the University of Chicago at the turn of the 20th century, and Artin's contribution was part and parcel of the revolution in algebra which took place in the 1920s.

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[1]) All rings are assumed to be *associative* unless otherwise stated.

The merit of a theory often rests on the efficacy of its concepts. Among the concepts of the theory of rings whose origin will be indicated in this article are the following: algebra, ring, module, division ring (algebra), semi-simple ring, nilpotent ring, simple ring, ideal, radical, quotient ring, direct sum, tensor product, chain conditions, idempotent and nilpotent elements, the Peirce decomposition.

A word about nomenclature. The subject we are dealing with did not begin as a study of "rings". In fact, the term "ring" was introduced by Hilbert in 1897, and then only in the concrete setting of rings of algebraic integers, which are commutative rings.[1]) The first abstract definition of a ring was given by Fraenkel in 1914 (see [31]). What we now call "ring theory" was known in the 19th century and in the first decades of the 20th century as the theory of "complex number systems" or "hypercomplex number systems" or (mainly in the United States) as the theory of "linear associative algebras". These systems consisted of elements of the form $\sum_{i=1}^{n} a_i e_i$, where a_i were initially real or complex numbers, and e_i formed a "basis" closed under multiplication, and which satisfied various other properties. Nowadays we call such objects "algebras". Matrices and group algebras are important examples.

The evolution of noncommutative ring theory will be outlined under the following headings:

I. *Sources*

(a) Symbolical algebra (Peacock, De Morgan *et al.*); quaternions (Hamilton); 1830-1843.

(b) Lie groups and Lie algebras (Lie, Scheffers, Killing, Poincaré, E. Cartan); 1880s-1890s.

II. *Exploration*

Cayley numbers, exterior algebras (Grassmann), group algebras (Cayley), matrices (Cayley), Clifford algebras, nonions (Sylvester); 1840s-early 1880s.

III. *Classification*

(a) Low-dimensional algebras (B. Peirce, Study, Scheffers); 1870s-1880s.

(b) Division algebras (Frobenius, C. S. Peirce); 1878, 1881.

(c) Commutative algebras (Weierstrass, Dedekind); 1884.

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[1]) The notion of *commutative* ring, under the name of "order", was defined by Dedekind in 1871 in the context of his work in algebraic number theory.

IV. *Structure of algebras*

(a) Over the complex and real numbers (Molien, E. Cartan, Frobenius); 1890s-1903.

(b) Over any field (Wedderburn); 1907.

V. *Interlude*

(a) Division algebras (Wedderburn, Dickson); 1905-1926.

(b) Definitions of abstract algebras and abstract rings (Dickson, Fraenkel); 1903, 1914.

VI. *Structure of rings with minimum condition*

The Artin-Wedderburn structure theorems (Artin); 1927.

VII. *Some subsequent developments*

(a) Deep study of division rings (algebras) (Albert, Brauer, Hasse, Noether); late 1920s-early 1930s.

(b) Nilpotent rings; 1930s-.

(c) Quasi-Frobenius rings (Nakayama); 1939-1941.

(d) Primitive rings (Jacobson); 1945.

(e) Prime rings (Goldie); 1958-1960.

(f) Representations of rings and algebras (Köthe, Brauer *et al.*); 1930s-.

(g) Homological methods (H. Cartan, Eilenberg, MacLane *et al.*); 1950s-.

We now turn to a point by point discussion of the above outline.

I. Sources

The early sources—symbolical algebra and quaternions—provided the impetus for the birth and early growth of the theory of hypercomplex number systems, while the later sources—Lie groups and Lie algebras—supplied the inspiration, motivation, and techniques for its mature development, resulting in the basic structure theorems for such systems (beginning with IV above). These two sources also reflected two distinct approaches to the subject of hypercomplex systems, two traditions—the abstract Anglo-American tradition and the more "down-to-earth" concrete Continental tradition, respectively (see [66]). We shall deal with the later sources (Lie groups and Lie algebras) when they begin to make their impact (under item IV).

In a strict sense, noncommutative ring theory originated from a single example, namely the quaternions created by Hamilton in 1843. These are "numbers" of the form $a + bi + cj + dk$ (a, b, c, d real numbers) which are added componentwise and multiplied according to the rules $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, with the obvious extension to all quaternions. This was the first example of a noncommutative "number system" (a system closed under the operations of addition, subtraction, multiplication and, initially, also division), and was the catalyst for the introduction of various other noncommutative number systems. We will consider some of these, as well as Hamilton's motivation for his creation, shortly. We wish to turn briefly now to an earlier development, namely the creation of symbolical algebra in the 1830s, which brought about, through speculations on the nature of algebra, a favourable climate for the construction of various "nontraditional" number systems.

The isolation of British mathematics in the 18th century had adverse effects on its development. The universities, for example, taught Newton's method of fluxions rather than the more powerful methods developed on the continent by the Bernoullis, Euler, and Lagrange. In the early 19th century several British mathematicians at Cambridge, among them Whewell, Peacock, and De Morgan, undertook a reform in the teaching of mathematics. In the British universities of this period mathematics was viewed more as an instrument for the training of logical minds than as a tool for the solution of practical problems. If this was to be effective, one would have to put algebra (which at the time consisted of symbolical manipulation relating to numbers) on a sound logical basis. That meant *justifying* the laws of operation with numbers (mainly negative and complex numbers). For example, why is $(-a)(-b) = ab$? Is $\sqrt{ab} = \sqrt{a}\sqrt{b}$ true for negative as well as positive numbers? Why is $a(b-c) = ab - ac$ when $b < c$? Why is $a^m a^n = a^{m+n}$ when m and n are negative or rational numbers?

The applicability of the "laws" of arithmetic to positive integers was considered "obvious". (For example, the equality $a(b-c) = ab - ac$ was not questioned when $b > c$.) The "science" of operations on the symbols representing positive integers was called "arithmetical algebra", and, as mentioned, was taken for granted. Against this, Peacock, De Morgan, Gregory, and others created "symbolical algebra", which was intended to have much broader applicability. We focus on Peacock's work, which had the greatest impact. It was embodied in his major publication *A Treatise on Algebra* of 1830.

Peacock wrote this work, he says, "with a view of conferring upon Algebra the character of a demonstrative science". To him symbolical algebra was "the science which treats of the combination of arbitrary signs and symbols by means of defined though arbitrary laws". As for the laws, he claims that

> We may assume any laws for the combination and incorporation of such symbols, so long as our assumptions are independent, and therefore not inconsistent with each other.

This, of course, is a rather modern point of view, well ahead of its time. This programme, however, was not implemented by Peacock in the stated generality. For although Peacock seems to profess freedom in choosing the laws which such a symbolical algebra can possess, he in fact postulates these laws to be the same as the laws of arithmetic. Thus, for example, the equality $a(b-c) = ab - ac$ is *decreed* to be true when $b < c$. We will not go into details here since the subject has been dealt with extensively in the literature. See [10], [15], [51], [60], [61], [68], [70], [71], [73].

To summarize, the doctrine of the British School of symbolical algebra, although not fully implemented, suggested that what matters in algebra are the rules which symbols obey rather than the meaning which one may attach to such symbols.[1]) Whatever its limitations, it provided a positive climate for subsequent developments in algebra. Thus symbolical algebra can be said to have given mathematicians the licence to create nontraditional number systems, while Hamilton and others showed them how to do it. In the words of Bourbaki [13]:

> The algebraists of the English school were the first to isolate, between 1830 and 1850, the abstract notion of law of composition, and then immediately broadened the field of algebra by applying this notion to a host of new mathematical entities: the algebra of logic (Boole), ... quaternions (Hamilton), matrices and nonassociative algebras (Cayley).

Hamilton, in his creation of quaternions in 1843, would seem to have practiced what Peacock had preached. This is not the case, however. Hamilton was, indeed, concerned with the foundations of algebra. As he remarked in 1853 (see [35]):

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[1]) This notion, that symbols have a "life of their own", without requiring recourse to meaning, has its precursor in the Operational Calculus of Lagrange and the work of others in the 18th century. It has been suggested that the inspiration for symbolical algebra came from these sources. See [51].

> The difficulties which so many have felt in the doctrine of Negative and Imaginary Quantities in Algebra forced themselves long ago on my attention.

Hamilton's proposed solution of these difficulties, however, was different from that of Peacock and his school of symbolical algebra. In particular, Hamilton objected to Peacock's view of the symbols of algebra as arbitrary marks without any *meaning*. To Hamilton the symbols of algebra had to stand for something "real"—not necessarily material objects but at least mental constructs. It was necessary, Hamilton claimed, to "look beyond the signs to the things signified." [1]) (Isn't this a foreshadowing of the formalist—intuitionist debate of almost a century later?)

Hamilton formulated his "philosophy" of algebra in an 1835 paper entitled "Theory of Conjugate Functions, or Algebraic Couples; With a Preliminary and Elementary Essay on Algebra as the Science of Pure Time." He was stimulated to write this essay, he says, by passages in Kant's *Critique of Pure Reason*. Briefly, just as (according to Kant) geometry is grounded in a mental intuition of space, so algebra should be grounded in a mental intuition of time. (For details see [35], [63], [68]). In this essay Hamilton also defines the complex numbers as ordered pairs of real numbers. Here he presents one of the earliest attempts to list systematically the properties of the real and complex number systems. He comes very close, in fact, to defining a field—the commutative and distributive laws are given, as are the closure laws, additive and mutiplicative inverses, and a definition of the zero element. What is missing are the associative laws—these he stated for the first time in his 1843 work on quaternions, to which we now turn.

Hamilton's motivation for the introduction of the quaternions is not easy to reconstruct. (For the *mathematical* thinking leading to their creation see [79].) His own (retrospective) view of them, given in 1855, is (see [35]):

> The quaternion [was] born, as a curious offspring of a quaternion of parents, say of geometry, algebra, metaphysics, and poetry... I have never been able to give a clearer statement of their nature and their aim than I have done in two lines of a sonnet addressed to Sir John Herschel:
>
> "An how the one of Time, of Space the Three. Might in the Chain of Symbols girdled be."

[1]) It is interesting that the mathematicians of the symbolical school soon came to think that Hamilton's work was well in accord with their own philosophy of algebra.

We might surmise that the geometric motivation was the desire to extend vectors in the plane to vectors in space (an extension which the quaternions, in a sense, accomplished); the algebra stemmed from a natural (from a mathematician's point of view) desire to extend number-pairs to triples, and, when this failed, to quadruples; the metaphysical connection with the ideas of Kant, which we mentioned above, was a factor in all of Hamilton's works in algebra; as for the poetry, we can do no better than to quote Weierstrass: "No mathematician can be a complete mathematician unless he is also something of a poet". See [35] for a detailed analysis.

For twenty two years following the invention of the quaternions, Hamilton was preoccupied almost exclusively with their application to geometry, physics, and elsewhere. To him the quaternions were the long-sought key which would unlock the mysteries of geometry and mathematical physics. The main importance of the quaternions, however, lay in another direction, namely in algebra (see [56]). Poincaré's tribute of 1902 is telling:

> Hamilton's quaternions give us an example of an operation which presents an almost perfect analogy with multiplication, which may be called multiplication, and yet it is not commutative... This presents a revolution in arithmetic which is entirely similar to the one which Lobachevsky effected in geometry.

We will explore some of the consequences of that revolution in the following section.

II. Exploration

Hamilton's quaternions at first received less than universal understanding and acclaim. Thus when Hamilton communicated his invention (discovery?) to his friend John Graves, the latter responded as follows [66]:

> There is still something in this system [of quaternions] which gravels me. I have not yet any clear view as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties.

Most mathematicians, however (including Graves) quickly came around to Hamilton's point of view. The floodgates were opened and the stage was set for the exploration of diverse "number systems", with properties which departed in various ways from those of the real and complex numbers.[1]) We give a brief outline of some of these.

(a) OCTAVES (Octonions, Cayley numbers) (1844)

Within three months of Hamilton's creation of the quaternions, John Graves outlined a system of "numbers" which he called "octaves". These are elements of the form $a_0 1 + a_1 e_1 + a_2 e_2 + \ldots + a_7 e_7$ (a_i real numbers), with basis elements $1, e_1, e_2, \ldots, e_7$ satisfying

$$e_i^2 = -1, 1 \cdot e_i = e_i \cdot 1, e_i e_j$$
$$= -e_j e_i (i \neq j), e_i e_{i+1} = e_{i+3}, e_{i+3} e_i = e_{i+1}, e_{i+1} e_{i+3} = e_i .$$

(See [50], [54], [80] for details.) This 8-dimensional algebra contains the quaternions, and is thus also noncommutative. It is, moreover, also not associative (i.e. $(ab)c \neq a(bc)$). It does possess, however, along with the quaternions, the property of unique division; that is, every nonzero element of the algebra has an inverse. We call such algebras *division algebras*; they will play an important role in our story.

The octaves (or octonions) are nowadays known as Cayley numbers since Cayley, independently of Graves, published a note on them in 1845. (It was a postscript to a short paper on elliptic functions.) Neither Graves nor Cayley pointed out the nonassociativity of these "numbers".

(b) TRIPLE ALGEBRAS (1844)

De Morgan was quickly converted to the point of view that it is legitimate to create systems whose properties diverge from those of the real numbers. In a paper of 1844 entitled "On The Foundations of Algebra" he introduced several algebras of triples of real numbers. These were commutative but nonassociative, and De Morgan designated them "imperfect".

(c) EXTERIOR ALGEBRAS (1844)

In his *Ausdehnungslehre* of 1844 Grassmann attempted to construct algebraically an abstract science of "spaces", freed from spatial conceptualiza-

[1]) As noted by Poincaré (p. 233), there is an analogy between the creation of quaternions and the creation of non-Euclidean (hyperbolic) geometry more than a decade earlier. Both achievements were radical violations of prevailing conceptions. Moreover, both inspired constructions of various analogous systems (algebras and geometries, respectively) which eventually led to their systematic classification (cf. secs. III & IV below for algebra and Klein's "Erlangen Program" for geometry).

tion and restriction to three dimensions—that is, to construct a vector algebra of n-dimensional space. The fundamental notion in his theory is that of an "extensive quantity", that is an expression of the form $a_1e_1 + \ldots + a_ne_n$, where $a_i \in \mathbf{R}$ (the real numbers) and e_i are linearly independent "units". On these quantities he defined various products, some of them yielding what are now known as exterior algebras. See [16], [19], [48], [50], [80] for details.

Grassmann's work was highly original but written obscurely and too abstractly to be appreciated at that time. A second, improved edition of his work, published in 1862, and a changing mathematical climate gained him the recognition he deserved.[1])

(d) Biquaternions (1853)

In his *Lectures on Quaternions* of 1853 Hamilton introduced "biquaternions", that is, "quaternions" with complex coefficients. He showed that they possess zero divisors (i.e. nonzero elements a and b such that $ab = 0$) and thus do not form a division algebra. In this work Hamilton also began consideration of "hypernumbers", namely n-tuples of real numbers.

(e) Group algebra (1854)

In 1854 Cayley published a paper entitled "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$", in which he defined a (finite) abstract group. At the end of this paper Cayley gave the definition of a *group algebra* (of a finite group over the real or complex numbers). He called it a system of "complex quantities" and observed that it is analogous in many ways to Hamilton's quaternions (i.e. it is associative, noncommutative, but, in general, not a division algebra). See [49], [80].

(f) Matrices (1855/1858)

In 1855, in a paper entitled "Remarques sur la notation des fonctions algébriques", Cayley introduced matrices, defined the inverse of a matrix and the product of two, and exhibited the relation of matrices to quadratic and bilinear forms. In an 1858 paper entitled "A memoir on the theory of matrices" he also defined the sum of matrices and the product of a matrix by a scalar, and showed (essentially) that $n \times n$ matrices form an associative algebra. In his own words [66]:

[1]) In this edition Grassmann explicitly mentions that his multiplication of extensive quantities applies, in particular, to yield the quaternions.

> It will be seen that matrices (attending only to those of the same order) comport themselves as *single quantities*; they may be added, multiplied or compounded together; the law of addition of matrices is precisely similar to that for the addition of ordinary algebraic quantities; as regards their multiplication (or composition), there is the peculiarity that matrices are not in general convertible [commutative].

In the same paper, Cayley shows that if L, M are 2×2 matrices such that $LM = -ML$, $L^2 = -1$, $M^2 = -1$, then letting $N = ML$ we get $N^2 = -1$, $MN = -NM$. He notes that L, M, N may serve as the i, j, k units of the quaternions, thus showing (essentially) that the quaternions are (isomorphic to) a subalgebra of the algebra of 2×2 matrices over $\mathbf{C}$ (the complex numbers).[1]) The relationship between hypercomplex number systems and matrices, of which this is an instance, was to be of fundamental importance for subsequent developments.

Cayley's introduction of matrices (as that of abstract groups around the same time) was well ahead of its time. (Frobenius, independently of Cayley, also discussed what amounts to the algebra of matrices, without using matrix notation, in a fundamental paper of 1878—see p. 240 below). Although Cayley appreciated the usefulness of matrices in simplifying systems of linear equations and composition of linear transformations, his broader concern for the abstract point of view in mathematics is apparent in much of his work. This was characteristic of a number of British mathematicians of the period, such as Peacock, De Morgan, Boole, Sylvester and, to some extent, Hamilton (who after his creation of the quaternions was at least partly converted to the symbolical-algebra point of view). See [41], [50], [66].

(g) CLIFFORD NUMBERS/ALGEBRA (1873/1878)

In a paper of 1873 entitled "Preliminary sketch of biquaternions," Clifford introduced, in connection with certain problems in geometry and physics, the so-called Clifford numbers (he called them "biquaternions", but they should be distinguished from Hamilton's biquaternions). These are elements of the form $q_1 + q_2\alpha$, where q_1, q_2 are quaternions, $\alpha^2 = 1$, and $\alpha q_i = q_i\alpha$. They form a nonassociative 8-dimensional algebra over $\mathbf{R}$ (not a division algebra).

In an 1878 paper entitled "Applications of Grassmann's extensive algebra", Clifford introduced what are known as "Clifford algebras"—associative $2n$-dimensional algebras, generated by n units $1, e_2, \ldots, e_n$, subject to the conditions $e_i^2 = -1$, $e_ie_j = -e_je_i$, and such that each product of two units is a new unit. See [48], [50], [80].

[1]) In 1854 Cayley showed that every finite (abstract) group is isomorphic to a group of permutations. See [49].

(h) Nonions (1882)

Sylvester wrote many papers between 1882 and 1884 on hypercomplex number systems, matrices, and their connections. (In fact, Sylvester claimed that he had created the theory of matrices independently of Cayley.) One such example was his system of "nonions"—a 9-dimensional algebra over **R**, generated by elements $u^i v^j (i, j = 0, 1, 2)$, where

$$u = \begin{pmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{pmatrix},$$

ρ being a cubic root of 1. He showed that this algebra is isomorphic to the algebra $M_3(\mathbf{R})$ of 3×3 matrices over **R**.

III. Classification

What we have presented above is only a sample, albeit a representative one, of various hypercomplex number systems introduced, largely by British mathematicians, within 30 or 40 years of Hamilton's work on quaternions. The scene now shifted to the United States and to continental Europe. Now that a stock of examples of noncommutative number systems had been established, one could begin to have a theory. The general concept of a (finite-dimensional) associative algebra (a hypercomplex number system) emerged, and there was a move to classify certain types of these general structures. We focus on three such developments.

(a) Low-dimensional algebras

Of fundamental importance here is B. Peirce's groundbreaking paper "Linear Associative Algebra" of 1870. In the last 100 pages of this 150-page paper Peirce classifies algebras (i.e. hypercomplex number systems) of dimension < 6 by giving their multiplication tables. There are, he shows, over 150 such algebras! What is important in this paper, though, is not the classification but the means used to obtain it.[1]) For here Peirce introduces concepts, and derives results, which proved fundamental for subsequent developments.

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[1]) Algebras of fixed (low) dimension were also classified, using different methods, by Scheffers, Study, and others. See [36], [38], [39], [66] for details. The complexity of the structure of general algebras, even of low dimensions, directed later researchers to focus on the study of special types of algebras (see e.g. (b) and (c) below and sec. IV).

Peirce's work is very much in the abstract Anglo-American tradition (cf. p. 229). Peirce was a great enthusiast of the quaternions and had taught them at Harvard as early as 1848. He was also an adherent of the symbolical approach to algebra. Algebra to Peirce was "formal mathematics", which was mathematics expressed by symbols which were not "tramelled by the conditions of external representation or special interpretation." In fact, Peirce's approach to mathematics in general was abstract, as can be seen from the "definition" of mathematics in the opening sentence of his treatise: "Mathematics is the science which draws necessary conclusions." This was certainly not a prevailing view of mathematics in the 19th century, although it is not unique to Peirce.[1])

Among the conceptual advances in Peirce's work are:

(1) An "abstract" definition of a *finite-dimensional associative algebra.* Peirce defines such an algebra (he calls it a "linear associative algebra") as the totality of formal expressions of the form $\sum_{i=1}^{n} a_i e_i$, where the e_i are basis elements.[2]) Addition is defined componentwise and multiplication by means of "structural constants" c_{ij}^k, namely $e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k$. Associativity under multiplication and distributivity are assumed, but not commutativity. This is probably the earliest conscious and explicit definition of an associative algebra (i.e. a hypercomplex number system).[3])

(2) The use of *complex coefficients.* Peirce takes the coefficients a_i in the expressions $\sum a_i e_i$ to be *complex* numbers. This conscious broadening of the field of coefficients from **R** to **C** (as we noted, both Hamilton and Clifford

[1]) Cf. the following "definitions" of mathematics expressing similar sentiments:

Gauss (1831): "Mathematics is concerned only with the enumeration and comparison of relations."

Grassmann (1844): "[Pure] mathematics is the science of forms."

Boole (1847): "It is not the essence of mathematics to be conversant with the ideas of number and quantity."

Hankel (1867): "[Mathematics is] purely intellectual, a pure theory of forms, which has for its objects not the combination of quantities or their images, the numbers, but things of thought to which there could correspond effective objects or relations, even though such a correspondence is not necessary."

Cantor (1883): "Mathematics is entirely free in its development and its concepts are restricted only by the necessity of being noncontradictory."

[2]) Peirce calls the basis elements the "alphabet" of the algebra. An algebra also has a "vocabulary" which consists of the operations of the algebra, as well as a "grammar" which gives the rules of composition (i.e. the postulates).

[3]) De Morgan, in the paper on foundations of algebra which we mentioned above, gave a similar, but less formal, description of such an algebra, calling it "a system of Algebra of the n^{th} character". Moreover, Grassmann in his *Ausdehnungslehre* of 1844, speakes of the "space" of "extensive quantities" (see (c) above).

presented *examples* of algebras with complex coefficients) was an important conceptual advance on the road to coefficients taken from an arbitrary field.

(3) *Identity* for the algebra is *not required.* This, too, is a departure from past practice and, again, gives an indication of Peirce's general, abstract approach. It made the statements and proofs of various results more difficult.

(4) Introduction of *nilpotent and idempotent elements.* An element x of an algebra is nilpotent if $x^n = 0$ for some positive integer n, idempotent if $x^2 = x$. These are two very important concepts which proved basic for the subsequent study of algebras (and later rings). After introducing these concepts[1]) Peirce proved the fundamental result that any algebra contains a nilpotent or an idempotent element. (Recall that the algebra need not have an identity.)

(5) The "Peirce decomposition". Peirce showed that if e is an idempotent of an algebra A then $A = eAe \oplus eB_1 \oplus B_2e \oplus B$, where $B_1 = \{x \in A : xe = 0\}$, $B_2 = \{x \in A : ex = 0\}$, and $B = B_1 \cap B_2$ ($\oplus$ indicates direct sum). This so-called Peirce decomposition of an algebra relative to an idempotent was a fundamental result which enabled Peirce to get a better hold on his algebra by studying its constituent parts. It is a central tool in the study of rings and algebras.

Peirce's work was well ahead of its time, and attracted little attention at first. Cayley, for example, who praised Peirce's work in an address in 1883 to the British Association for the Advancement of Science, called it "outside of ordinary mathematics"[2]). Even some of Peirce's admirers in the United States characterized the work as "philosophy of mathematics" rather than mathematics proper. Peirce, of course, turned out to have been a *mathematical* pioneer. See [62], [66], [69] for details.

(b) DIVISION ALGEBRAS

As we mentioned, the first example of a noncommutative algebra, namely Hamilton's quaternions, was a division algebra. The question arose as to which other systems of n-tuples of real numbers (hypercomplex number systems) possessed unique division (i.e. were division algebras). The answer was given, independently, by Frobenius (in 1878) and by C.S. Peirce (B. Peirce's son, in 1881), namely that the real numbers, the complex numbers, and the quater-

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[1]) Idempotent elements appeared in the work of Boole twenty years earlier.

[2]) This is ironic, coming from Cayley. His own work of 1854 on abstract groups was neglected by the mathematical community for twenty years! See [49].

nions are (in our terminology) the only possible finite-dimensional associative division algebras over **R**.

Frobenius' work appears at the end of a seminal paper entitled "Über lineare substitutionen and bilineare Formen", in which he develops the theory of matrices in the language of bilinear forms. (The forms, he says, can be viewed "as a system of n^2 quantities which are ordered in n rows and n columns.") In the final section of the paper Frobenius defines a hypercomplex number system (he calls it a "form system") as consisting of elements of the form $\sum_{i=1}^{m} a_i E_i (a_i \in \mathbf{R}$ *or* $\mathbf{C})$, where the E_i are *some* linearly independent bilinear forms in the variables $x_1, ..., x_n; y_1, ..., y_n$ such that the product of any two of them is again a linear combination of $E_1, E_2, ..., E_m$. The form systems, then, are subalgebras of $M_n(\mathbf{R})$ or $M_n(\mathbf{C})$. (As we mentioned before, the relationship between hypercomplex systems and matrices, noted here by Frobenius, will play an important role in subsequent developments.) "Especially remarkable", Frobenius says, "are such systems of real forms for which the determinant of $\sum_{i=1}^{m} a_i E_i$ cannot vanish for real values of $a_1, a_2, ..., a_m$ without all these coefficients being identically zero". Frobenius thus singles out here for special attention the real division algebras. He then asks and answers the (more or less obvious) question: What are all of the real division algebras?

C.S. Peirce's proof of the above result on real division algebras appeared in one of the many notes he added to his father's paper "Linear Association Algebra" which he (C.S.) published in the *American Journal of Mathematics* in 1881 [67]. (B. Peirce originally published 100 copies of his work, in lithographed form, and sent them to his friends and mathematical acquaintances.) C.S. Peirce's statement of the theorem reads: "Ordinary real algebra, ordinary algebra with imaginaries, and real quaternions are the only associative algebras in which division by finites [i.e. by nonzero elements] always yields an unambiguous quotient." [1]) See [48], [54], [66] for details.

(c) Commutative algebras

The result we have in mind here is that a finite-dimensional associative and commutative algebra over **R** or **C**, without nilpotent elements, is a direct sum

[1]) As we previously mentioned, the Cayley numbers (octaves) form an 8-dimensional real division algebra. It is, however, not associative, but is *alternative*: $(a^2)b = a(ab)$ and $a(b^2) = (ab)b$ for every a and b in the algebra. In 1950 E. Kleinfeld showed that (aside from the reals, complex numbers and quaternions) there are no other finite-dimensional alternative real division algebras (see [4], [54]). In 1958, Bott, Kervaire, and Milnor showed, using high-powered methods of differential topology, that the only finite-dimensional real division algebras over **R** (not necessarily alternative) have dimensions 1, 2, 4, or 8.

of a number of copies of either **R** or **C**. Thus one not only adds but also multiplies the elements of the algebra componentwise (when they are given as $\sum a_i e_i$). An immediate consequence of this result is that the only commutative division algebras over **R** are **R** or **C**. (This latter result also follows, of course, from that of Frobenius/Peirce in (b) above.)

The above characterization of commutative algebras over **R** and **C** was obtained, independently, by Weierstrass and Dedekind in the 1860s, although their works were published only in 1884-85 (see [80]). Both men were motivated, at least in part, by the following remark of Gauss, made in an 1832 paper on complex numbers [66]:

> The author [Gauss] has reserved for himself [the task] of working out more completely the subject, which in the present treatise is actually only occasionally touched upon. There then, too, the [following] question will find its answer: Why can the relations between things which present a multiplicity of more than two dimensions not furnish still other kinds of quantities permissible in the general arithmetic?

It is remarkable that Gauss seems to have anticipated here (as also, of course, in connection with fundamental developments in other branches of mathematics) the study of hypercomplex systems, and the fact that there are no systems analogous to **C** (i.e. fields) whose dimensions are greater than 2.

Dedekind's interest in commutative rather than general hypercomplex systems is understandable. In his fundamental work on ideal theory in algebraic number fields, Dedekind views the number field as an extension of the field of rational numbers, hence as a finite-dimensional algebra over the rationals. He exploits this point of view in his studies of algebraic number theory. To Dedekind, then, a finite-dimensional commutative algebra was a familiar object.

Dedekind's work helped to stimulate the deeper works of Molien and Cartan on the structure of more general types of algebras. This is part of the story to which we turn next.

IV. Structure of algebras

The first example of a noncommutative algebra was given by Hamilton in 1843. During the next forty years mathematicians introduced other examples of noncommutative algebras, began to bring some order into them and to single out certain types of algebras for special attention. Thus low-dimensional algebras, division algebras, and commutative algebras were classified and characterized. The stage was (almost) set for the founding of a general theory of finite-dimensional, noncommutative, associative algebras. The task was accomplished in the last decade of the 19th century and the first decade of the 20th century. Before that, however, important developments took place in a neighboring branch of mathematics which had an impact on the work in associative algebras. This was the founding of the theory of Lie groups and Lie algebras in the 1870s and 1880s, to which we shall shortly turn (see source (b) in the outline on p. 228 and the remark following on pp. 229-230). But first a description of the structure theory of associative algebras—the main subject of this section IV.

If A is a finite-dimensional associative algebra, we have the following result:

(a) $A = N \oplus B$, where N is nilpotent and B is semi-simple.[1] (An algebra N is *nilpotent* if $N^k = 0$ for some positive integer k. An algebra is *semi-simple* if it has no nontrivial nilpotent ideals—this, at least, was the initial conception of semi-simplicity.)

(b) $B = C_1 \oplus C_2 + \ldots \oplus C_n$, where C_i are simple algebras (i.e. have no nontrivial ideals).[2]

(c) $C_i = M_{n_i}(D_i)$, the algebra of $n_i \times n_i$ matrices with entries from a division algebra D_i.

The above representations are, moreover, unique (i.e. the n, n_i are unique, and the N, B, C_i, D_i are unique up to isomorphism.)

These results were derived for algebras over **R** and **C** by Molien, Cartan, and Frobenius, and about ten years later for algebras over an arbitrary field by Wedderburn. It should be noted that this type of structure theorem (i.e. the decomposition of a structure into "simple" substructures) was not new to algebra. Although the immediate inspiration and motivation for this result came from the neighboring theory of Lie algebras, there were other precedents.[3] In the Lie algebra case, Killing and Cartan made the major

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[1]) The decomposition of A into N and B does not hold for algebras over arbitrary fields although it holds over **R** & **C**. See [3] for the conditions under which it is true.

[2]) The nilpotent part N is intractable, even today.

[3]) The decomposition of an integer into a unique product of primes is the first such instance and goes back to Greek antiquity. In the 1870s Dedekind gave a decomposition of ideals in an algebraic number field into prime ideals. The decomposition of a finite abelian group into a direct product of cyclic groups of prime power order, given by Frobenius and Stickelberger in 1879, is another example.

contributions in the 1880s by decomposing the semi-simple Lie algebras into simple ones and then classifying the simple Lie algebras (see [18], [47]). Cartan was thus a major contributor to both the Lie and associative theories, and, in addition to the model, he also carried over some of the techniques from the Lie to the associative case. There were also *direct* connections between Lie *groups* and hypercomplex number systems, to which we turn shortly. But first a word about the origin of Lie groups and Lie algebras.

Lie founded the theory of continuous transformation groups (what we today call Lie groups) in the 1870s so as to facilitate the study of differential equations (cf. Galois theory). Just as Galois associated a finite (discrete) group of permutations with an algebraic (polynomial) equation, so Lie associated an infinite (continuous) group of transformations with a differential equation. Lie subsequently showed that for the purposes of the differential equation it suffices to focus on the "local" structure of the Lie group—that is, on the "infinitesimal transformations" which, when multiplied using the "Lie product", form a Lie algebra. (If S, T are infinitesimal transformations, so is their Lie product $[S, T]$ which is given by $[S, T] = ST - TS$.) Just as in the case of algebraic equations, so too in this theory the objects of special interest are the "simple" Lie groups. These give rise to "simple" Lie algebras. Lie thus proposed the task of studying the structure of Lie algebras with special attention to be given to simple Lie algebras. This task (as we mentioned) was admirably accomplished by Killing and Cartan.

In an influential two-page paper of 1882 called "Sur les nombres complexes" Poincaré highlighted the connection between Lie groups and hypercomplex systems:

> The remarkable works of M. Sylvester on matrices have recently drawn attention to complex numbers [hypercomplex number systems] analogous to Hamilton's quaternions. The problem of complex numbers is easily reduced to the following: to find all the continuous groups of linear substitutions in n variables, the coefficients of which are linear functions of n arbitrary parameters.

Poincaré had in mind the following: If H is a hypercomplex system, then every element $u \in H$ determines a linear transformation $u_R : H \to H$ given by $u_R(x) = xu$. Poincaré then wanted to determine all possible continuous groups of such transformations; in modern terms, he was interested in finding all subgroups of the Lie group $GL_n(\mathbf{C})$ of invertible $n \times n$ matrices over $\mathbf{C}$. (Note that u_R, if invertible, can be thought of as an element of $GL_n(\mathbf{C})$.) Thus a hypercomplex system gives rise to a Lie group. Conversely, given an

(n-parameter) Lie group $G = \{T_u : u = (u_1, \ldots, u_n)$ is a parameter$\}$ (the elements of a Lie group were given as continuous transformations

$$x_i' = f_i(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_n)$$

of n variables x_i, and n parameters $u_i, i = 1, 2, \ldots, n$), the multiplication $T_u T_v = T_w$ in G determines a product $uv = w$ of the parameters (n-tuples). The question is: when does this represent multiplication in a hypercomplex number system? [1]) See [18], [27], [40], [47], [80] for details.

Of the three mathematicians—Molien, Cartan, Frobenius—who gave the structure theorems for associative algebras over the real and complex fields, Cartan was the most influential, so we shall focus (briefly) on his contributions. For details see [40], [66], [80].

Cartan's work on the subject appeared in an 1898 paper entitled "Sur les groupes bilinéaires et les systèmes de nombres complexes". As the title indicates, the paper deals with the relation between hypercomplex systems and Lie groups (see the comments above on this relationship). The thrust and major part of the paper, however, are directed to a development of the structure of hypercomplex systems *independent of Lie group theory*. Both Scheffers and Molien had previously obtained many of the results which now appeared in Cartan's work on the structure of such systems. (Cartan was apparently unaware of Molien's work. For details of both Scheffers' and Molien's contributions see [66].) Scheffers and Molien, however, had used techniques and results from the Lie theory to study the associative case. It was Cartan's expressed aim to avoid this. As he says, his development of the structure theory of associative algebras "does not call forth any notion from the theory of [Lie] groups and stays exclusively in the realm of the theory of [hyper]complex numbers." Cartan's main contribution, then, lies in the development of methods and concepts *internal* to the theory of hypercomplex number systems.

[1]) The association of one type of mathematical structure with another so as to study the latter by means of the former is, of course, a very fruitful idea in mathematics. We have already noted the association of Lie groups to hypercomplex systems, Lie groups to Lie algebras, differential equations to Lie groups, algebraic equations to finite groups. Other examples are the associations between equations and curves (analytic geometry), groups and fields (Galois theory), groups and geometries (Klein's Erlangen Program), topological spaces and homology groups (algebraic topology), groups and matrices (representation theory). Moreover, any associative algebra can be turned into a Lie algebra by defining in it a new product, the so-called Lie (or bracket) product $[x, y] = xy - yx$. Conversely, any Lie algebra can be embedded in an associative algebra, the so-called Birkhoff-Witt enveloping algebra of a Lie algebra, in which the Lie product corresponds to the product in the Lie algebra. We thus have an association between Lie algebras and associative algebras.

Cartan associates a characteristic and minimal polynomial with each *algebra* (see e.g. [3])—these are fundamental tools in his development of the theory. Their factors are related to the structure of the given algebra.[1]) Thus the underlying vector space of the associative algebra, and hence the fields of scalars **R** and **C**, play a fundamental role in Cartan's work. Not only his proofs but also some of his concepts rely on the linear algebra structure. For example, Cartan defines a "pseudo-null" element of an algebra as one whose characteristic polynomial has only the zero root. It can be shown that this notion is equivalent to that of a nilpotent element (see [66], p. 297), defined almost thirty years earlier by Benjamin Peirce, although Cartan does not make that identification. Cartan's proofs, then, are not conceptual and are often quite long (e.g. some five-page proofs of Cartan can be replaced by five-line proofs in the style of Wedderburn, whose work we shall describe shortly). What proved lasting in Cartan's work in addition to the main structure theorem, though he apparently did not attach to them great significance, were the four concepts which he introduced at the end of this development in order to *state* his structure theorem more succinctly. These were the notions of direct sum, two-sided ideal,[2]) and simple and semi-simple algebra. (Others before Cartan used these concepts, but only implicitly.) For example, Cartan defines an ideal (he calls it an "invariant subsystem") as follows:

> We say that a system Σ admits an invariant subsystem σ, if every element of σ belongs to Σ and if the product, on the right or on the left, of an arbitrary element of σ and an arbitrary element of Σ belongs to σ.

Cartan's description of simple algebras (which he defined as those having no invariant subsystems) is as follows:

> All simple systems of complex numbers are of the same kind; they are formed by p^2 basis elements e_{ij}, where i and j take on all values $1, 2, \ldots, p$ and the law of multiplication of these basis elements is given by the formulas $e_{ij}e_{jl} = e_{il}$, $e_{ij}e_{\lambda l} = 0 (j \neq \lambda)$. The number p is an

[1]) Recall the well-known result from linear algebra that if V is a finite-dimensional vector space and $T: V \to V$ a linear transformation with characteristic polynomial $f(t) = (t-\lambda_1)^{m_1} \ldots (t-\lambda_r)^{m_r}$, then $V = V_1 \oplus \ldots \oplus V_r$, where $V_i = \{x \in V: (T-\lambda_i I)^n x = 0$ for some $n\}$. This gives an indication of the kind of ideas involved.

[2]) Dedekind introduced the notion of an ideal for rings of algebraic integers in the 1870s, in connection with his fundamental work in algebraic number theory. Kronecker dealt with ideals (which he called "divisors" and "modular systems") of polynomial rings in 1882. Both of these types of rings are, of course, commutative. There is, in any case, no mention in Cartan's work of the ideals of Dedekind or Kronecker.

arbitrary integer greater than or equal to 1. These systems are called p^2-ions.[1])

Wedderburn's work (1907)

At the end of the 19th century the theory of hypercomplex number systems had attained a degree of maturity. All-important connections had been made with Lie's theory of continuous groups (as we have seen), as well as with the theory of finite groups, via group representation theory.[2]) At the same time a major structure theorem was available, established without recourse to external theories. Partly because of this, and partly because of the connections with other major fields, the theory of hypercomplex systems became a distinct discipline for serious mathematical investigation (on both sides of the Atlantic). What was needed now was a new departure. This was provided by Wedderburn's groundbreaking paper of 1907 entitled "On hypercomplex numbers" [82].

It has been said that a "good abstract theory" is one which summarizes and unifies previous results, placing them in a new perspective, and one which provides new directions for subsequent work in the field. Without doubt, Wedderburn's work qualifies as such.[3])

The major result in Wedderburn's paper, namely the structure theorem for finite-dimensional associative algebras, is essentially the same as that given by Cartan (see p. 242). There was "merely" an extension of the field of scalars of the algebra from **R** and **C** to an arbitrary field. This extension, however, necessitated a new approach to the subject—a rethinking and reformulation of the major concepts and results of the theory of hypercomplex number systems.

Wedderburn came to the University of Chicago from Scotland in 1904 as a Carnegie Research Fellow. Here he met E.H. Moore, Bolza, Maschke, and Dickson, all of whom provided great inspiration for the 22-year-old Wedderburn. This very strong school of algebra at the University of Chicago excelled

[1]) It is interesting that the language of matrices, introduced in the 1850s, was not very familiar even as late as 1898.

[2]) A basic tool in the study of group representations is the group algebra of a finite group. This is a semi-simple hypercomplex system (at least over **R** and **C**) to which the structure theorem of Cartan can therefore be applied. These ideas were fundamental in the development of group representation theory. See [40] for details.

[3]) Other outstanding examples of "good abstract theories" in algebra are Steinitz' theory of fields of the 1910s (see footnote on p. 253), E. Noether's theory of commutative rings with the ascending chain condition of the 1920s, and Artin's formulation of Galois theory in the 1930s.

in the abstract Anglo-American tradition, following in the footsteps of such luminaries as Boole, Cayley, B. Peirce, and C.S. Peirce (see [8], [65]). This was, indeed, fertile ground for the young Scotsman.

Wedderburn clearly recognized that his major contributions lay in the *methods* he was developing to prove the structure theorems. In his own words:

> The object of this paper is in the first place to set the theory of hypercomplex numbers on a rational basis.[1]) The methods usually employed in treating the parts of the subject here taken up are, as a rule, dependent on the theory of the characteristic equation and for this reason often valid only for a particular field or class of fields. Such, for instance, are the methods used by Cartan...

We now give a brief summary of some of Wedderburn's main concepts, methods, and results.

(1) The notion of *ideal* is central to the study. Although, as we noted, the concept of ideal appears in Cartan's work (and, to some extent, also in Molien's and Frobenius'), it is only with Wedderburn that it is given a central place in the study of algebras. (Recall that Cartan defines ideals towards the *end* of his paper so that he can *state* his results succinctly.) "The theory of invariant sub-algebras [ideals]", Wedderburn says, "is of great importance." In fact, it forms the essence of the "rational basis" upon which Wedderburn founds his work, replacing the "auxiliary" characteristic and minimal polynomials of Cartan.

(2) The concept of *difference (quotient) algebra* is defined. Wedderburn mentions the analogy with the quotient group of a finite group, which was introduced in 1889 (see [49]).[2]) Among other results, he shows that if B is a maximal invariant subalgebra of A, then A/B is a simple algebra.

(3) The concept of *nilpotent algebra* is introduced. "Nilpotent algebras are of great importance in the discussion of the structure of algebras", Wedderburn notes. (Recall Cartan's "pseudo-null" [nilpotent] elements which he defined in terms of the roots of the characteristic polynomial.) Wedderburn

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[1]) That is, independent of the field of scalars. In fact, toward the end of the paper Wedderburn says that "It is remarkable that the properties of a field with regard to division are not used in many of the theorems of the preceding sections".

[2]) The notion of quotient structure is implicit in Gauss' congruences modulo n of 1801. This idea was modelled by Cauchy in 1847 in his definition of the complex numbers as congruence classes of real polynomials modulo $x^2 + 1$. Kronecker, in connection with his work in algebraic number theory and algebraic geometry in the 1880s, generalized Cauchy's device to quotient rings of polynomial rings (in any number of indeterminates) with respect to their ideals.

shows that every algebra contains a maximal nilpotent invariant subalgebra (it may, of course, be zero) which contains all other nilpotent invariant subalgebras. This is what we today call the *radical* of the algebra (if the algebra if finite-dimensional), although Wedderburn does not have a special name for it. (The term "radical" is due to Frobenius—it appears in his 1903 work on the structure of hypercomplex number systems. The *notion* of radical was adumbrated earlier by Molien and Cartan.)

(4) A *semi-simple algebra* is defined as today. Cartan and others *defined* a semi-simple algebra as a direct sum of simple algebras. For Wedderburn, "Algebras which have no nilpotent invariant sub-algebra form a very important class. Such algebras are called *semi-simple*."

Wedderburn shows that if one "factors out" the radical of an algebra one gets a semi-simple algebra. "This theorem", he claims, "is very important, its importance lying in the fact that... it enables us to confine our attention to algebras which have no nilpotent invariant subalgebra." This technique was central to his successful study of the structure of algebras. By factoring out the radical and focusing on the well-behaved semi-simple part of the algebra, Wedderburn was able to arrive efficiently and conceptually at his main results, "whereas CARTAN microscopically dissected his algebra and found himself entangled in the extremely complicated structure of the radical" (Parshall [66]).

Among the results which Wedderburn establishes for a semi-simple algebra A are the following:

(i) A is a direct sum of simple algebras.
(ii) If e is an idempotent element of A then eAe is semi-simple.
(iii) A contains an identity element.

(5) Explicit recognition is given to the concept of a *division algebra*. Since in the past algebras were considered only over **R** or **C**, the concept of a division algebra was of no special consequence. (As we have noted, there is only one (finite-dimensional) noncommutative division algebra over **R**, namely the quaternions, and there are none over **C**—as Wedderburn shows in this paper; see sec. V below). Wedderburn's structure theorems for algebras over an arbitrary field highlight, however, the concept of a division algebra. Moreover, as Wedderburn says, "[The structure theorems are] incomplete in so far as the classification is given in terms of primitive [division] algebras which have themselves not yet been classified."

Wedderburn defines a division algebra as one having a single idempotent element and no nilpotent elements, and then shows that in such an algebra every nonzero element is invertible. He also shows that if e is a primitive idempotent of an algebra A (he defines e to be "primitive" if it is the only idempotent in A—this is equivalent to the modern definition; see [3]), then eAe is a division algebra.

(6) The *tensor product* of algebras is introduced. Although Scheffers made use of this notion in 1891, Wedderburn was the first to define formally and explicitly the important notion of the tensor product of two algebras:

> If C and D are any algebras such that every element of the one is commutative with every element of the other, and if the order of the complex [subspace] $A = CD$ is the product of the orders of C and D, then A is an algebra which is called the *direct* [tensor] *product* of C and D.

Wedderburn then states his main theorem on the classification of simple algebras in terms of the tensor product:

> Any simple algebra can be expressed as the direct [tensor] product of a primitive [division] algebra and a simple matric algebra.

Wedderburn also proves the converse of this theorem, namely:

> The direct product A of a primitive algebra B and a quadrate matric algebra C is simple.

Wedderburn called the algebra of $n \times n$ matrices over a field a "simple or quadrate matric algebra of order n^2."

(7) The *Peirce decomposition* of an algebra is extended. For Wedderburn, just as for his predecessor Peirce, idempotents play a central role in the study of algebras. Wedderburn extends the Peirce decomposition of an algebra (see p. 239) by the use of (what we call today) pairwise orthogonal primitive idempotents (see [3], [46]), and is thus able to minutely dissect the algebra.

Eighty years after Wedderburn's work his concepts, methods, and results are still basic to our study of algebras (but see sec. VII (g) concerning homological methods). Moreover, Wedderburn's paper can easily be read by today's student. (By contrast, reading, say, Cartan's work—only ten years earlier—on the subject is a formidable task.)

Wedderburn provided, indeed, a truly "rational basis" for the study of the structure of algebras. By bringing into relief the important concepts and methods of the theory, his work invited the generalization by Artin to rings with minimum condition (see sec. VI below). What it left open is important questions about the structure of nilpotent algebras and division algebras (see secs. V, VII). It is perhaps fitting to conclude our discussion of Wedderburn's work with several tributes:

> [Wedderburn] was the first to find the real significance and meaning of the structure of a simple algebra. This extraordinary result has excited the fantasy of every algebraist and still does so in our day. (Artin, [6].)

> Wedderburn's pioneering work on the structure of simple algebras set the stage for the deep investigations—often with an eye to applications in algebraic number theory—in the theory of algebras. (Herstein, [43].)

> [Wedderburn's] rational methods struck at the heart of the theory of algebras, and their influence is felt even to this day... His work neatly and brilliantly placed the theory of algebras in the proper, or at least in the modern, perspective. (Parshall, [66].)

V. INTERLUDE

The title is not meant to suggest a lack of activity in the study of algebras during the first two decades or so of the 20th century. There were simply no *fundamental* developments in the period between the work of Wedderburn in 1907 and the works of Artin, Noether *et al.* in the 1920s. Below we briefly describe two areas of progress in these intervening years.

(a) DIVISION ALGEBRAS

As we noted, Wedderburn's structure theorems left unresolved the nature of division algebras. Knowledge of finite-dimensional division algebras over a field F was available only in the following three cases:

(i) $F = \mathbf{R}$. In this case, as we have seen, there are only three division algebras over F, namely the reals, complex numbers, and quaternions.

(ii) $F =$ an algebraically closed field (eg. $\mathbf{C}$). In this case Wedderburn himself showed (in the 1907 paper where his structure theorem appears) that over such a field there are no division algebras except for the field itself. As Wedderburn put it:

> If the given field is so extended that every equation is soluble, the only primitive [division] algebra in the extended field is the algebra of one unit, $e = e^2$.

(iii) F = a finite field. Since the algebra is finite-dimensional it must, in fact, be finite in this case. Wedderburn had earlier (1905) established the remarkable result that in such a case the division algebra is commutative (see [81]).[1]) The division algebra over the finite field F is thus itself a finite field. Still earlier (1893) E. H. Moore had characterized all finite fields. This case, then, was also completely solved.

No other examples of division algebras were known at that time (in 1905). Thus the only "genuine" (i.e. noncommutative) division algebra known was the quaternions. Dickson noted that the discovery and classification of division algebras was the chief outstanding problem in the theory of algebras over an arbitrary field. He then proceeded (in 1905 and 1914) to contribute to its solution by exhibiting new classes of division algebras over an arbitrary field (see [26] for details). He also showed that there are infinitely many nonisomorphic quaternion algebras over the rationals (see [4]). In the 1920s Dickson (and others) defined the important class of *cyclic* division algebras. (See footnote on p. 257 for a definition.) The major step in the study of division algebras was the description, in the early 1930s, of all division algebras over the field of *rational* numbers (see sec. VII).

(b) DEFINITIONS OF AN ABSTRACT ALGEBRA AND AN ABSTRACT RING

The definition of an "associative algebra" (hypercomplex number system) throughout the 19th century, and even in the early 20th century (for example, in Wedderburn's 1907 paper) was that of a system of elements of the form $\sum a_i e_i$ (a_i elements of a field, e_i "basis" elements), with componentwise addition and with multiplication of the basis elements given by "structural constants", obeying certain laws (see p. 238). In 1903 Dickson gave the first more or less abstract definition of an associative algebra [25]. To him, it is a

> System of elements $A = (a_1, a_2, \ldots, a_n)$ each uniquely defined by n marks of the field F together with their sequence. The marks $a_1, \ldots, a_n$ are called the *coordinates* of A. The element $(0, 0, \ldots, 0)$ is called zero and designated 0. Addition of elements is defined thus:
> $A + B = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.
> It follows that there is an element $D = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$ such that $D + B = A$.
> Consider a second rule of combination of the elements having the properties:

[1]) The theorem, for example, provides the only known proof that in a finite projective plane Desargues' theorem implies Pappus' theorem. Artin [6] claims that this theorem of Wedderburn "has fascinated most algebraists to a very high degree."

1. For any two elements A and B of the system, $A \cdot B$ is an element of the system whose coordinates are bilinear functions of the coordinates of A and B, with fixed coefficients belonging to F.
2. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, if $A \cdot B, B \cdot C, (A \cdot B) \cdot C, A \cdot (B \cdot C)$ belong to the system.
3. There exists in the system an element I such that $A \cdot I = A$ for every element A of the system.
4. There exists in the system at least one element A such that $A \cdot Z \neq 0$ for any element $Z \neq 0$.

Dickson then goes on to show that "any system of elements given by [this] definition is a system of complex numbers according to the usual ... definition." This he does by first *proving* the distributive law and the uniqueness of the identity element I. Dickson then proves the independence of these postulates.

This "abstract" definition of an associative algebra (a coordinate-free, entirely abstract definition of an algebra was given by Dickson in 1923—see [28]) was one instance of a general interest by American mathematicians around this time in abstract, postulational definitions of algebraic systems and, in particular, in establishing the independence of the postulates of such systems. For example, definitions of groups were given (between 1901 and 1905) by Huntington, E.H. Moore, Dickson, and Pierpont, and of fields (in 1903) by Dickson and Huntington. These definitions of groups and fields were entirely abstract (even from our point of view).[1] See [8], [10], [12], [49].

In 1914, Fraenkel, in a paper entitled "On zero divisors and the decomposition of rings" [31], was the first to give an abstract definition of a *ring* (cf. p. 2). He gave diverse examples of the concept he was defining, which included both commutative and non-commutative rings, namely integers modulo n, hypercomplex number systems, matrices, and p-adic integers. It was an abstract definition in today's style. Thus Fraenkel defines a ring as "a system" with two (abstract) operations, to which he gives the names addition and multiplication. Under one of the operations (addition) the system forms a group (he gives its axioms). The second operation (multiplication) is associative and distributes over the first operation. Two axioms give the closure of the system under the operations, and there is the requirement of an identity in the definition of the ring. Commutativity under addition does *not* appear as an axiom but is proved!; so are other elementary properties of

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[1]) Somewhat earlier (at the end of the 19th Century) we witness the emergence of an abstract, axiomatic approach in geometry (Pasch, Peano, Hilbert) and arithmetic (Dedekind, Frege, Peano).

a ring such as $a \times 0 = 0$, and $a(-b) = (-a)b = -(ab)$. There are two extraneous axioms (dealing with "regular" elements in the ring) which depart from an otherwise modern definition.

Among the main concepts introduced are "zero divisors" and "regular elements". Fraenkel deals in this paper only with rings which are not integral domains and discusses divisibility for such rings. Much of the paper deals with decomposition of rings as direct products of "simple" rings (not the usual notion of simplicity).

Fraenkel's aim in this paper was to do for rings what Steinitz had just (1910) done for fields, namely to give an abstract and comprehensive theory of (commutative and noncommutative) rings.[1]) Of course he was not successful (he does admit that the task here is not as "easy" as in the case of fields)—it was too ambitious an undertaking to try to subsume the structure of both commutative and noncommutative rings under one theory. Fraenkel did, however, delineate the abstract notion of a ring and, in this respect, made a significant contribution.

VI. Structure of rings with minimum condition

In a fundamental paper of 1927 entitled "Zur Theorie der hyperkomplexen Zahlen" [5], Artin proved a structure theorem for rings with minimum condition (descending chain condition)[2]) which generalized Wedderburn's structure theorem for finite-dimensional algebras (discussed in sec. IV). The theorem, now known as the Wedderburn-Artin theorem for semi-simple rings with minimum condition (i.e. rings without nilpotent ideals and satisfying the descending chain condition for, say, right ideals—see e.g. [43]) states that if R is such a ring, then it is a direct sum of simple rings and these, in turn, are matrix rings over division rings; moreover, the above representations are unique (cf. Wedderburn's structure theorem, p. 246).

As we note, the *result* is essentially the same as Wedderburn's. It is, however, the spirit of the work and the conceptual advances which make it

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[1]) Steinitz' "Algebraische Theorie der Körper" of 1910 was the first abstract study of fields as a distinct subject. This fundamental work, which some say marked the beginning of modern abstract algebra, arose out of a desire to delineate the abstract notions common to the various contemporary theories of fields. It provided the basic concepts of field theory necessary for the subsequent abstract study of Galois theory, algebraic number theory, and algebraic geometry.

[2]) Artin proved his theorem for rings satisfying both the ascending and descending chain conditions. Later (1939) Hopkins showed that the descending chain condition suffices.

stand out as a very important contribution. Artin's work, though, must be seen against the background of the revolution in algebra which was taking place in the 1920s. It was initiated by Emmy Noether in a paper in 1921 on commutative rings with the ascending chain condition (now called Noetherian rings) entitled "Idealtheorie in Ringbereichen." We see here the beginnings of the conceptual, abstract, axiomatic approach to algebra. This is where the spirit (if not all the content) of so-called modern algebra originated.

Of course, E. Noether was not the first to use abstraction in algebra. Earlier (late 19th century) Dedekind had employed it in ideal theory and Galois theory; Frobenius, Weber *et al.* in group theory; and in the early 20th century, Wedderburn in associative algebra theory, and Steinitz in field theory. What Noether did was to bring unity and conceptual clarity to those developments. She highlighted what was essential in past work in abstract algebra by creating or bringing into prominence a number of central concepts and revealing hitherto unnoticed connections. To her abstract algebra was a distinct, conscious discipline, with its own concepts, methodology, and basic results. And, of course, the proof of her success was in the fertility of her approach, which animated not only algebra but also other branches of mathematics, such as topology, analysis, and number theory (eg. algebraic topology, functional analysis, and class-field theory, respectively). In Noether's own words [24]: "Algebra is the foundation and tool of all mathematics."

We briefly highlight three of her fundamental accomplishments. These still guide algebraic thinking today. Thus Noether

(a) Recognized the importance of *chain conditions*. In two great memoirs (in 1921 and 1927) on ideal theory, Noether founded the abstract study of rings with chain conditions.[1]) In the first (see above) she gave an abstract treatment of the decomposition theories of Hilbert, Lasker, and Macaulay for polynomial rings, and in the second (entitled "Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern") an axiomatic treatment of the theories of Dedekind and Kronecker for algebraic number and function fields. As Bourbaki noted [13]:

> It is seen in these memoirs how a small number of abstract ideas, such as the notion of irreducible ideal, the chain conditions, and the idea of an integrally closed domain ... can by themselves lead to general results

[1]) The ascending chain condition was introduced by Dedekind in connection with his study of ideals in an algebraic number field. Wedderburn, in his 1907 paper on the structure of algebras, uses "descending chain condition" arguments, without employing that term.

which seemed inextricably bound up with results of pure computation in the cases where they had previously been known.

(b) Gave prominence to the concept of *module*. Although this concept had been used earlier by Dedekind in concrete settings, Noether was the first to define it abstractly and note its importance as a unifying concept in algebra. In particular, she showed the usefulness of viewing representations as modules, which resulted in the absorption of the theory of group representations into the study of modules over rings and algebras (see [22]). Unlike earlier methods, this approach applied to fields of arbitrary characteristic. Noether also stressed that both ideal theory and the structure theory of algebras can be viewed as applications of module theory. This module-theoretic point of view, enabling the "linearization" of problems, has, of course, become fundamental.

(c) Highlighted the concept of *ring*. As we have noted, the concept of a ring was introduced in concrete settings by Dedekind and Hilbert and in the abstract by Fraenkel. It was Noether, however, who, through her groundbreaking papers, in which the concept of ring played a fundamental role, brought this concept into prominence as a central concept of algebra, taking its rightful place alongside those of group and field. The concept of ring immediately began to serve as the starting point for much of the development of abstract algebra that followed.

Let us conclude our account of Noether's work with several testimonials:

> Emmy Noether was one of the most influential mathematicians of this century. ...The development of abstract algebra, which is one of the most distinctive innovations of 20th century mathematics, is largely due to her. (N. Jacobson, in the introduction to E. Noether's collected works.)
>
> She taught us to think in simple and thus general terms ... homomorphic image, the group or ring with operators, the ideal ... and not in complicated algebraic calculations; and she therefore opened up a path to the discovery of algebraic regularities where before these regularities had been obscured by complicated specific conditions. (P. Alexandroff [15].)
>
> The methodological concepts of arithmetization, generalization, abstraction, reduction, and transfer are the spindles she used to trim and combine in an orderly fashion the algebraic threads that had been generated, separated, and entangled with geometric and analytic strands during the preceding century. (U. Merzbach [75].)

Artin was himself a major contributor to the concepts, methods, and results of abstract algebra during the crucial decade of the 1920s. His structure theorem for rings with the descending chain condition is, in fact, a model in the spirit of this period. The features to note are:

(a) The *ring* rather than the algebra becomes the central object of study. Noether began to prepare the ground (with her 1921 paper) for the ascendancy of the concept of ring. Her work, together with Wedderburn's conceptual treatment of his structure theorems for algebras, made it "natural" (in the hands of a master like Artin) to extend the theorems to rings (with minimum condition). The concept of ring now becomes central.

(b) *Chain conditions* acquire prominence. In Noether's papers of 1921 and 1927 the *ascending* chain conditions for ideals is the central notion. In Artin's work the *descending* chain condition is introduced for the first time and acquires importance.

(c) *One-sided ideals* are used as an essential tool. Artin's rings, contrary to Noether's, are noncommutative. Here the concept of one-sided ideal, briefly introduced in Wedderburn's 1907 paper (as "semi-invariant subalgebra"), acquires central importance. In fact, the chain conditions in Artin's paper apply to one-sided ideals.

Artin's theorem proved to be a model and an essential tool in subsequent work on the structure of rings. In the words of Herstein [34]:

> [The Wedderburn-Artin structure theorem] is the cornerstone of many things done in algebra. From it comes out the whole theory of group representations. In fact there are very few places in algebra—at least where noncommutative rings are used—where it fails to make its presence felt.

The new ideas in abstract algebra of Artin, Noether, and others were disseminated in 1930-31 in an influential book by Van der Waerden entitled "Modern Algebra" [77]. G. Birkhoff [10] gives an absorbing account of its impact on the wider mathematical community:

> Even in 1929, its concepts and methods [i.e. of "modern" algebra] were still considered to have marginal interest as compared with those of analysis in most universities, including Harvard. By exhibiting their mathematical and philosophical unity and by showing their power as developed by Emmy Noether and her other younger colleagues (most notably E. Artin, R. Brauer, and H. Hasse), van der Waerden made "modern algebra" suddenly seem central in mathematics. It is not too much to say that the freshness and enthusiasm of his exposition electrified the mathematical world—especially mathematicians under 30 like myself.

VII. Some subsequent developments

Cartan's structure theorems of the end of the 19th century can be said to have brought to an end the first phase in the evolution of the structure theory of noncommutative rings (algebras), Wedderburn's theorems of 1907 the second phase, and Artin's results of 1927 the third. Artin's work ushered in the next phase, still with us today, in which his work served as a model, a guide, and an inspiration for various subsequent developments in noncommutative ring theory. Thus in (a) and (b) below (see outline on pp. 2-3) we deal with two basic questions left open in the Wedderburn-Artin structure theorems, namely the nature of division rings and nilpotent rings, respectively. In (c), (d), and (e) we discuss generalizations of the Wedderburn-Artin theorems obtained by weakening or removing one or both of its two conditions (semi-simplicity and the minimum condition); this gives rise to quasi-Frobenius rings, primitive rings, and prime rings, respectively. In (e) we comment on the neighbouring area of representations of rings and algebras which, as Herstein noted (see p. 256), derives from the Wedderburn-Artin structure theorems, and in (f) we touch on a fundamental "new" method—homological algebra. We now very briefly sketch these developments.

(a) Division rings (algebras)

We first want to point out that the study of division rings is coextensive with that of division algebras, since the centre of a division ring is a field, and hence the division ring can be viewed as a division algebra over its centre.

The Wedderburn-Artin structure theorems leave unresolved the nature of division algebras over an arbitrary field. Over the fields **R**, **C**, and a finite field, all finite-dimensional division algebras were known by the end of the first decade of the 20th century (see pp. 250-251). In the next two decades, Dickson, Wedderburn, and others introduced new classes of division algebras, especially the important class of cyclic division algebras.[1] The major problem tackled and completely solved during the late 1920s and early 1930s was the classification of division algebras over the rationals and, more generally, over an algebraic number field. The result, known as the Albert-Brauer-Hasse-Noether theorem, is that all such division algebras are cyclic. Jacobson [48] called this

[1]) An algebra A over a field F is *cyclic* if there exists a nonzero element $u \in A$ such that $u^n \in F$ for some positive integer n, and if there exists a maximal subfield K of A which is invariant under the inner automorphisms induced by u, and such that $1, u, \ldots, u^{n-1}$ form a K-basis of A. See e.g. [3] or [4].

result "one of the most important achievements of algebra and number theory in the 1930s." A central tool in the study of these algebras, invented by Brauer in 1929, is the "Brauer group" of a field. The proof also involved deep arithmetic properties of algebraic number fields. See [3], [80] for details.

The problem of the classification of (even finite-dimensional) division algebras is far from solved. Intense research to understand their structure is going on nowadays, using such high-powered tools as K-theory and étale cohomology. See [46] and the recent books [23] and [30] on the subject.

(b) Nilpotent rings

The Wedderburn-Artin structure theorems deal with algebras and rings which are nilpotent-free (semi-simple). At the end of his paper of 1907 on the structure of algebras Wedderburn notes that "the classification of algebras cannot be carried much further than this till a classification of nilpotent algebras has been found..." Attempts had been made in the past to deal with the problem (see e.g. [42]), but, to this day, the results are fragmentary (see e.g. [52]).

(c) Quasi-Frobenius rings

The class of quasi-Frobenius rings is one of the most interesting classes of non-semi-simple rings. A ring R is *quasi-Frobenius* if R satisfies the minimum condition on (say) right ideals and if $rl(J) = J$ and $lr(L) = L$ for every right ideal J and left ideal L of R, where for any subset S of R, $l(S) = \{x \in R : xS = 0\}$, $r(S) = \{x \in R : Sx = 0\}$.

One can show that a semi-simple ring with minimum condition is quasi-Frobenius, hence quasi-Frobenius rings are generalizations of the rings studied by Artin in his structure theorem. In fact, many of the results which can be proved for semi-simple rings with minimum condition are also true for quasi-Frobenius rings, but not for arbitrary rings with minimum condition. That is one reason for studying quasi-Frobenius rings. A second, and the original, reason derives from the theory of group representations. Frobenius introduced the so-called Frobenius algebras around the turn of this century in that context (see [22]), and Nakayama later (1939-41) generalized these to quasi-Frobenius rings and algebras.

In the theory of group representations, Maschke's theorem is of fundamental importance. It states that if G is a finite group and F a field whose characteristic does not divide the order of G, then the group algebra $F(G)$ is semi-simple (and, of course, satisfies the minimum condition, since G is finite).[1] This result, Nakayama showed, generalizes as follows: If G is finite, F *any* field, then $F(G)$ is quasi-Frobenius. In fact, the following even more general result holds: If R is any quasi-Frobenius ring, G any finite group, then the group ring $R(G)$ is quasi-Frobenius. This is the representation-theoretic context of quasi-Frobenius rings.

Quasi-Frobenius rings have also been generalized, and these rings with their generalizations form the subject of much current research interest. See [76] for details.

(d) PRIMITIVE RINGS

In the 1940s Jacobson arrived at an important extension of the Wedderburn-Artin structure theorems to rings *without* minimum condition, the so-called semi-primitive and primitive rings. The basic problem was to find the "right" definition of the "radical" of a ring—the previous definition of the radical as the maximal nilpotent ideal not being applicable since the ring no longer satisfied the minimum condition. The problem is not trivial. As Herstein notes [43]:

> The aim of defining the radical [is] to concentrate the bothersome behaviour of a ring in a piece of it such that when this piece [is] removed the resulting ring [is] well enough behaved to permit some delicate dissection.

This Jacobson did extremely well. The proof was, of course, in the fruitful results obtained by employing this radical.[2]

A ring is called *semi-primitive* if its (Jacobson) radical is zero. (For rings with minimum condition the notions of semi-primitivity and semi-simplicity coincide.) The other basic notion defined by Jacobson was that of "primitive ring", which was a generalization of simple ring.[3] Given these definitions, the structure theorem states that:

(a) a semi-primitive ring is a subdirect sum of primitive rings.

---

[1]) It is also known that $F(G)$ is semi-simple for any group G (not necessarily finite) and a non-countable field F of characteristic zero. It is not known, however, if $F(G)$ is semi-simple for countable F.

[2]) One definition of the Jacobson radical is as the intersection of the maximal left (or right) ideals of the ring. There are many equivalent descriptions. See [46], [48] for details. See also [33] for various other radicals.

[3]) A ring is called (left) *primitive* if it contains a maximal left ideal which contains no nonzero ideals of the ring. Again, there are many equivalent descriptions (see [46]). In the presence of the minimum condition, primitivity and simplicity of a ring are equivalent concepts.

(b) a primitive ring is isomorphic to a "dense ring of linear transformations" of a vector space over a division ring. (A "dense ring of linear transformations" is a certain subring of the ring of linear transformations in an infinite-dimensional vector space over a division ring. For example, the ring of row-finite matrices is such a ring.)

This structure theorem of Jacobson was a far-reaching generalization of the Wedderburn-Artin structure theorem. See e.g. [43], [48] for details.

(e) PRIME RINGS

Quasi-Frobenius rings are non-semi-simple rings. Primitive rings do not satisfy the minimum condition. Thus in each of these two cases one of the two conditions in the rings which Artin studied—semi-simple rings with minimum condition—is eliminated. In the present case of prime rings both semi-simplicity and the minimum condition are replaced by weaker conditions. This provides another very important extension, obtained by Goldie in 1958-60, of the Wedderburn-Artin theorems, to semi-prime (and prime) rings with maximum condition. (As we noted, in the presence of an identity in the ring, the minimum condition implies the maximum condition.)

A ring is *semi-prime* if it has no nonzero nilpotent ideals. (For rings with minimum condition this is equivalent to semi-simplicity.) A ring is *prime* if $AB = 0$ implies $A = 0$ or $B = 0$, where A and B are ideals of the ring. Armed with these definitions, Goldie states the structure theorem as follows:

(a) A ring R is semi-prime with maximum condition if and only if it has a classical quotient ring $Q(R)$ which is a semi-simple ring with minimum condition.[1])

(b) A ring R is prime with maximum condition if and only if R has a quotient ring $Q(R)$ which is simple with minimum condition (i.e., by the Wedderburn-Artin theorem, if and only if R can be "tightly embedded" in an $n \times n$ matrix ring over a division ring—a type of "dense embedding" of R in a finite matrix ring, just as a primitive ring can be "densely embedded" in an "infinite matrix ring", that is, in a ring of linear transformations of an infinite dimensional vector space). See [43], [46] for details.

Goldie's theorem assumes the same all-important role in the study of rings with the ascending chain condition (Noetherian rings) which the Wedderburn-

[1]) (i) The maximum condition is equivalent to the ascending chain condition. Rings, not necessarily commutative, satisfying this condition are often called Noetherian rings. (ii) Goldie demands chain conditions somewhat weaker than the maximum condition. (iii) The "classical quotient ring" of a ring is an extension of the notion of the "field of quotients" of an integral domain.

Artin theorem assumes in the study of rings with the descending chain condition (Artinian rings). In fact, the study of noncommutative Noetherian rings, stimulated by Goldie's work, has been an active area of research ever since. One of the major problems is to be able to "do" algebraic geometry in noncommutative Noetherian rings, and thus to extend such notions as localization, divisibility, and torsion to the noncommutative case. See for example [20], [32].

(f) REPRESENTATIONS OF RINGS AND ALGEBRAS

The theory of group representations—a very important device in the study of finite groups—was developed by Burnside, Frobenius, and Molien towards the end of the 19th century (see [40]). It was soon found that the group algebra of the group over an appropriate field was an important tool in the study of the representations of the group. (The group algebra is semi-simple and hence the structure theorems of Cartan and Wedderburn can be applied.) In fact, the study of the representations of a group can easily be shown to be reducible to the study of the representations of the group algebra of the given group. Thus attention turned to representations of algebras and then (after the Artin structure theorem) to those of rings. In a fundamental paper in 1929, Noether had shown the conceptual advantages of a shift of focus—from representations of algebras and rings to modules over these algebras and rings, respectively. This is, in large measure, the point of view taken today.

Major departures following the work of Noether were taken by Brauer and Köthe in the 1930s. The major problem is the decomposition of modules over rings into "simple" components. What these simple components are, which modules decompose, which rings give rise to a decomposition of all modules over these rings, are some of the basic questions asked. New classes of rings arose from these studies, among them quasi-Frobenius rings (as we have seen) and uniserial rings. This is a very active field of current interest. See [34], [72], [76] for some of the recent work, and [22] for an account of the "classical" theory.

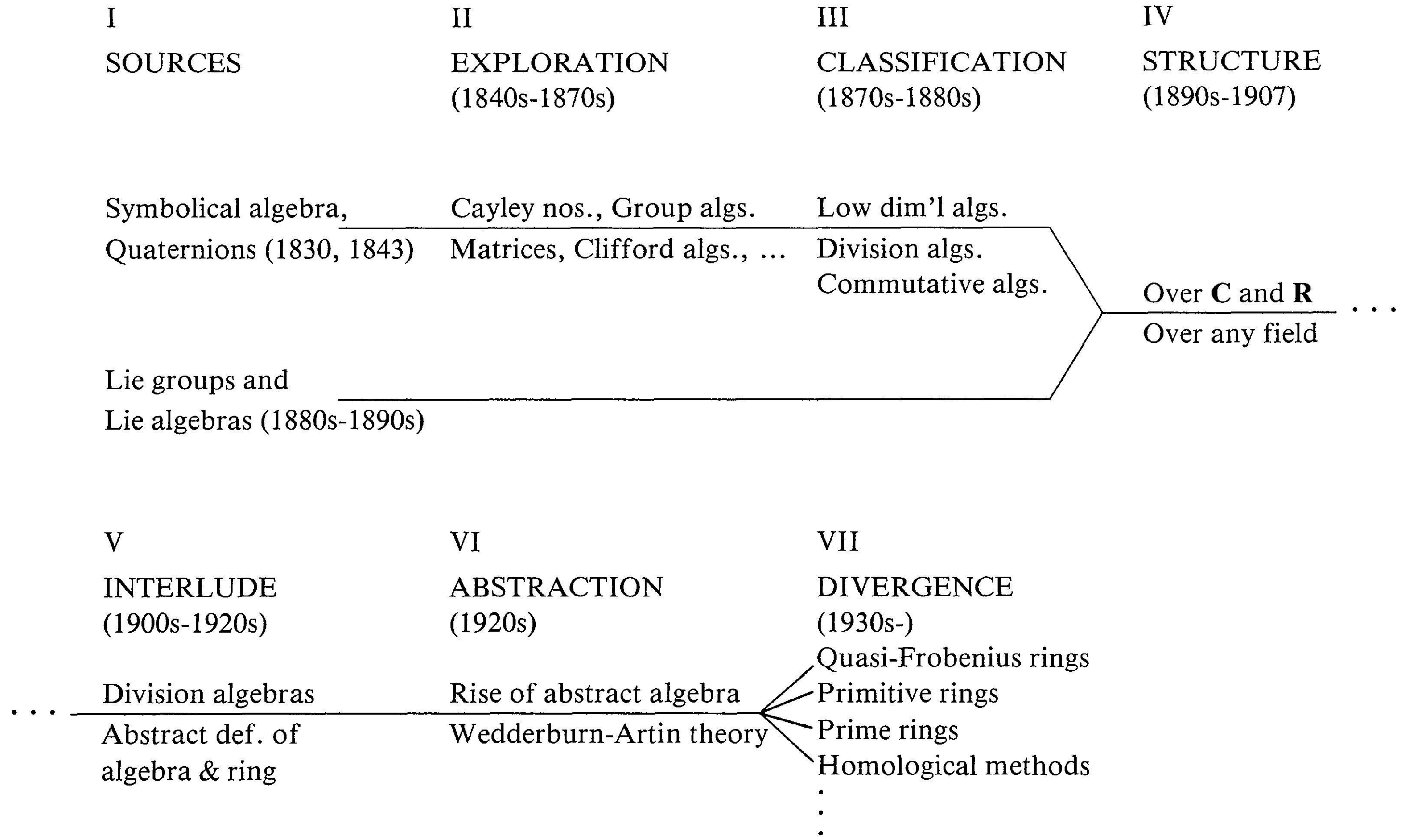
(g) HOMOLOGICAL METHODS

Homological algebra is an offspring of algebraic topology, the latter, in turn, being inspired by the developments in abstract algebra and the suggestions initiated by Noether in the 1920s (see [57]). Among its fundamental concepts are the functors *Ext* and *Tor* which, in a sense, measure the manner in which modules over general rings "misbehave" when compared to the "nice" vector spaces of classical linear algebra. Other basic concepts of homological algebra are those of projective and injective module. (Every module is a homomorphic image of a projective module and a submodule of an injective module.) See [15] for details.

The debt that (algebraic) topology owed to algebra has been amply repaid. Homological (and category theory) methods have invaded much of abstract algebra, and especially ring theory—both commutative and noncommutative—beginning with the 1950s. It suffices to compare standard textbooks, let alone research monographs, of the 1950s (and even the 1960s) with those of the 1970s and 1980s (e.g. Jacobson's texts [45] and [48] to note the fundamental differences in language and technique. In fact, many of the standard concepts and results have been rephrased in homological language. For example, R is a division ring if and only if every R-module is free: R is semisimple and Artinian if and only if every R-module is injective, if and only if every R-module is projective, if and only if R has zero global (homological) dimension; R is quasi-Frobenius if and only if R satisfies the minimum condition and is injective as an R-module.

Finally, it might be of interest to mention that in addition to homological methods, analytic and topological methods have also invaded the study of rings; in fact, they have given rise to new branches of mathematics such as normed rings and differential rings (cf. Lie groups and topological groups). Another field related to that dealt with in this article is that of nonassociative rings; there are three important classes of such rings, giving rise to distinct theories, namely Lie rings, Jordan rings, and alternative rings. See e.g. [53].

We conclude this article with a diagrammatic sketch of the evolution of noncommutative ring theory as outlined in the various sections.


I
SOURCES
II
EXPLORATION
(1840s-1870s)
III
CLASSIFICATION
(1870s-1880s)
IV
STRUCTURE
(1890s-1907)
Symbolical algebra,
Quaternions (1830, 1843)
Cayley nos., Group algs.
Matrices, Clifford algs., ...
Low dim'l algs.
Division algs.
Commutative algs.
Over C and R
Over any field
Lie groups and
Lie algebras (1880s-1890s)
V
INTERLUDE
(1900s-1920s)
VI
ABSTRACTION
(1920s)
VII
DIVERGENCE
(1930s-)
Division algebras
Abstract def. of
algebra & ring
Rise of abstract algebra
Wedderburn-Artin theory
Quasi-Frobenius rings
Primitive rings
Prime rings
Homological methods

REFERENCES

Most of the references given here are to *secondary* sources. Extensive references, especially to *primary* sources, may be found in [3], [13], [29], [40], [61], [66], and [80].

[1] ABIAN, A. *Linear Associative Algebra*. Pergamon Press, 1971.

[2] ALBERT, A. A. *Modern Higher Algebra*. Cambridge University Press, 1938.

[3] —— *Structure of Algebras*. Amer. Math. Soc., 1939.

[4] —— (ed). *Studies in Modern Algebra*. Math. Assoc. of America, 1963.

[5] ARTIN, E. Zur Theorie der hyperkomplexen Zahlen. *Hamb. Abh. Math. Sem. 5* (1927), 251-260.

[6] —— The influence of J.H.M. Wedderburn on the development of modern algebra. *Bull. Amer. Math. Soc. 56* (1950), 65-72.

[7] ARTIN, E., C. NESBITT and R.M. THRALL. *Rings with Minimum Condition*. Univ. of Michigan Press, 1944.

[8] BELL, E.T. Fifty years of algebra in America, 1888-1938. *Amer. Math. Soc. Semi-centennial Addresses, Vol. II,* A.M.S., 1938.

[9] —— *The Development of Mathematics*. McGraw-Hill, 1945 (2nd ed.).

[10] BIRKHOFF, G. Current trends in algebra. *Amer. Math. Monthly 80* (1973), 760-782.

[11] —— (ed.). Proceedings of American Academy workshop on the evolution of modern mathematics. *Hist. Math. 2* (1974), 425-624.

[12] —— (a) The rise of modern algebra to 1936 and (b) The rise of modern algebra, 1936 to 1950. In: *Men and Institutions in American Mathematics,* ed. by D. Tartwater *et al.,* Texas Tech. Press, 1976, pp. 41-63 and 65-85.

[13] BOURBAKI, N. *Eléments d'Histoire des Mathématiques*. Hermann, 1969.

[14] BREWER, J.W. and M.K. SMITH. *Emmy Noether: A Tribute to her Life and Work*. Marcel Dekker, 1981.

[15] CARTAN, H. and S. EILENBERG. *Homological Algebra*. Princeton Univ. Press, 1956.

[16] CHEVALLEY, C. *Fundamental Concepts of Algebra*. Academic Press, 1956.

[17] CLOCK, D.A. A New British Concept of Algebra: 1825-1850. PhD Dissertation, Univ. of Wisconsin, 1965.

[18] COHEN, A. *An Introduction to the Lie Theory of One Parameter Groups*. D.C. Heath & Co., 1911.

[19] COLLINS, J.V. An elementary exposition of Grassmann's "Ausdehnungslehre". *American Math. Monthly,* a series of articles in Vols. *6* (1899) and *7* (1900).

[20] COZZENS, J.H. and F.L. SANDOMIERSKI. *Noncommutative Ring Theory*. Lecture Notes in Mathematics No. 545, Springer-Verlag, 1975.

[21] CROWE, M.J. *A History of Vector Analysis*. Univ. of Notre-Dame Press, 1967.

[22] CURTIS, C.W. and I. REINER. *Representation Theory of Finite Groups and Associative Algebras*. J. Wiley & Sons, 1962.

[23] DAUNS, J. *A Concrete Approach to Division Rings*. Heldermann Verlag, 1982.

[24] DICK, A. *Emmy Noether: 1882-1935*. Birkhäuser, 1981.

[25] DICKSON, L. E. Definition of a linear associative algebra by independent postulates. *Trans. Amer. Math. Soc. 4* (1903), 21-26.
[26] —— *Linear Algebras.* Cambridge Tracts in Mathematics No. 16, Hafner, 1971 (orig. 1914).
[27] —— On the relation between linear algebra and continuous groups. *Bull. Amer. Math. Soc. 22* (1915), 53-61.
[28] —— *Algebras and their Arithmetics.* Dover, 1960 (orig. 1923).
[29] DIEUDONNÉ, J. *Abrégé d'Histoire des Mathématiques, 1700-1900, Vol. I.* Hermann, 1978.
[30] DRAXL, P. K. *Skew Fields.* Cambridge Univ. Press, 1983.
[31] FRAENKEL, A. Über die Teiler der Null und die Zerlegung von Ringen. *Jour. für die Reine und Angew. Math. 145* (1914), 139-176.
[32] GOLAN, J. S. *Localization of Noncommutative Rings.* Marcel Dekker, 1975.
[33] GRAY, M. *A Radical Approach to Algebra.* Addison-Wesley, 1970.
[34] GUSTAFSON, W. H. The history of algebras and their representations. In: *Representations of Algebras,* ed. by M. Auslander & E. Lluis, Springer Lecture Notes in Math., 1982, pp. 1-28.
[35] HANKINS, T. L. *Sir William Rowan Hamilton.* The Johns Hopkins Univ. Press, 1980.
[36] HAPPEL, D. Klassifikationstheorie endlich-dimensionaler Algebren in der Zeit von 1880 bis 1920. *L'Enseignement Math. 26* (1980), 91-102.
[37] HARKIN, D. The development of modern algebra. *Norsk Mat. Tidskr. 33* (1951), 17-26.
[38] HAWKES, H. On hypercomplex number systems. *Trans. Amer. Math. Soc. 3* (1902), 312-330.
[39] —— Enumeration of non-quaternion number systems. *Math. Ann. 58* (1904), 361-379.
[40] HAWKINS, T. Hypercomplex numbers, Lie groups, and the creation of group representation theory. *Arch. Hist. Ex. Sc. 8* (1971/72), 243-287.
[41] —— The theory of matrices in the 19th century. In: *Proc. Intern. Congr. Math.* (Vancouver), 1974, pp. 561-570.
[42] HAZLETT, O. On the classification and invariantive characterization of nilpotent algebras. *Amer. Jour. Math. 38* (1916), 109-138.
[43] HERSTEIN, I. N. *Noncommutative Rings.* The Carus Math. Monographs No. 15, Math. Assoc. of Amer., 1968.
[44] JACOBSON, N. *The Theory of Rings.* Amer. Math. Soc., 1943.
[45] —— *Lectures in Abstract Algebra, 3 Vols.* D. Van Nostrand Co., 1951, 1953, and 1964.
[46] —— *Structure of Rings.* Amer. Math. Soc., 1956.
[47] —— *Lie Algebras.* John Wiley & Sons, 1962.
[48] —— *Basic Algebra I, II.* W. H. Freeman & Co., 1974 and 1980.
[49] KLEINER, I. The evolution of group theory: a brief survey. *Math. Mag. 59* (1986), 195-215.
[50] KLINE, M. *Mathematical Thought from Ancient to Modern Times.* Oxford Univ. Press, 1972.
[51] KOPPELMAN, E. The calculus of operations and the rise of abstract algebra. *Arch. Hist. Ex. Sc. 8* (1971/72), 155-242.
[52] KRUSE, R. L. and D. T. PRICE. *Nilpotent Rings.* Gordon & Breach, 1969.
[53] KUROSH, A. G. The present status of the theory of rings and algebras. *Uspekhi Matem. Nauk. 6* (1951), 3-15.

[54] ––– *Lectures on General Algebra*. Chelsea Publ. Co., 1963.
[55] LICHTENBERG, D. R. The Emergence of Structure in Algebra, PhD Dissertation, The Univ. of Wisconsin, 1966.
[56] MACDUFFEE, C. C. Algebra's debt to Hamilton. *Scripta Math. 10* (1944), 25-35.
[57] MACLANE, S. History of abstract algebra. In: *American Mathematical Heritage: Algebra and Applied Mathematics,* ed. by D. Tarwater *et al.,* Texas Tech. Press, 1981, pp. 3-35.
[58] MCCOY, N. H. *The Theory of Rings*. Macmillan, 1964.
[59] MERZBACH, U. C. Quantity to Structure: Development of Modern Algebraic Concepts from Leibniz to Dedekind, PhD Dissertation, Harvard Univ., 1964.
[60] NOVY, L. L'Ecole algébrique anglaise. *Revue de Synthèse 89* (1968), 211-222.
[61] ––– *Origins of Modern Algebra*. Noordhoff Intern. Publ., 1973.
[62] ––– Benjamin Peirce's concept of linear algebra. *Acta hist. rerum natur. necnon techn., Special Issue 7* (1974), 211-230.
[63] ØHRSTRØM, P. W. R. Hamilton's view of algebra as the science of pure time and his revision of this view. *Hist. Math. 12* (1985), 45-55.
[64] PARSHALL, K. H. In pursuit of the finite division algebra theorem and beyond: Joseph H. M. Wedderburn, Leonard E. Dickson, and Oswald Veblen. *Arch. Intern. d'Hist. des Sc. 33* (1983), 274-299.
[65] ––– Eliakim Hastings Moore and the founding of a mathematical community in America, 1892-1902. *Ann. of Sc. 41* (1984), 313-333.
[66] ––– Joseph H. M. Wedderburn and the structure theory of algebras. *Arch. Hist. Ex. Sc. 32* (1985), 223-349.
[67] PEIRCE, B. Linear Associative Algebra. *Amer. Jour. Math. 4* (1881), 97-215. (Contains comments by C. S. Peirce.)
[68] PYCIOR, H. M. The Role of Sir William Rowan Hamilton in the Development of British Modern Algebra, PhD Dissertation, Cornell Univ., 1976.
[69] ––– Benjamin Peirce's *Linear Associative Algebra. Isis 70* (1979), 537-551.
[70] ––– George Peacock and the British origins of symbolical algebra. *Hist. Math. 8* (1981), 23-45.
[71] ––– Internalism, externalism, and beyond: 19th century British algebra. *Hist. Math. 11* (1984), 424-441.
[72] SHOFIELD, A. H. *Representations of Rings over Skew Fields*. Cambridge Univ. Press, 1985.
[73] SMITH, G. S. De Morgan and the laws of algebra. *Centaurus 25* (1981), 50-70.
[74] SPITTISWOODE, W. Remarks on some recent generalizations of algebra. *Proc. London Math. Soc. 4* (1871-73), 147-164.
[75] SRINIVASAN, B. and J. SALLY (eds.). *Emmy Noether in Bryn Mawr*. Springer-Verlag, 1983.
[76] TACHIKAWA, H. *Quasi-Frobenius Rings and Generalizations*. Lecture Notes in Mathematics, Springer-Verlag, 1973.
[77] VAN DER WAERDEN, B. L. *Modern Algebra*. Springer-Verlag, 1930-31.
[78] ––– Die Algebra seit Galois. *Jahresbericht der Deutsch. Math. Ver. 68* (1966), 155-165.
[79] ––– Hamilton's discovery of quaternions. *Math. Mag. 49* (1976), 227-234.
[80] ––– *A History of Algebra, from al-Khwarizmi to Emmy Noether*. Springer-Verlag, 1985.

[81] WEDDERBURN, J. H. M. A theorem on finite algebras. *Trans. Amer. Math. Soc.* *6* (1905), 349-352.

[82] —— On hypercomplex numbers. *Proc. London Math. Soc.* *6* (1907), 77-118.

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