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Artikel: SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS

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Anhang: Appendix: an improvement of the inequality of Theorem 1

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It is clear that these two arcs of $\tilde{a}$ pass $2i-1$ times from right to left under a. Thus the contribution of the neighbourhood $\mathcal{U}$ to $Q(\alpha, \alpha)$ is given by

$$\sum_{i=1}^{n} (2i-1) = -n + 2\sum_{i=1}^{n} i = n^2 .$$

This shows that $Q(\alpha, \alpha) > 0$ if a crosses at least one band of V. If not, then $\alpha = 0$.

Thus Q is positive definite. This completes the proof of Theorem 2.

Appendix: an improvement of the inequality of Theorem 1

Though the inequality

$$(10) \qquad c(K) + r(K) - 1 \geqslant \operatorname{span}(L)$$

of Theorem 1 becomes an equality for weakly alternating diagrams, it may be sharpened a little for other cases. Let K be a link diagram in R^2 and let $\Gamma \subset R^2$ be the associated link projection. For $P \in S^2 - \Gamma$ (where $S^2 = R^2 \cup \{\infty\}$), let $i(P)$ be the intersection number modulo 2 of Γ with a generic 1-chain connecting P to ∞. Shade the regions of $S^2 - \Gamma$ for which $i \equiv 1 \pmod 2$, so that S^2 is painted like a chessboard. Let $b_1, ..., b_m$ be the shaded regions of $S^2 - \Gamma$ and let $w_1, ..., w_n$ be the unshaded regions of $S^2 - \Gamma$.

An edge e of Γ is called *K-good* either if e is a loop or if one of the end points of e corresponds to an overcrossing point of K and the other end point of e corresponds to an undercrossing point of K. An edge of Γ which is not K-good is called *K-bad.* For any $i \in \{1, ..., m\}$ and for any $j \in \{1, ..., n\}$, it is clear that the set $\overline{b_i} \cap \overline{w_j}$ consists of several edges and double points of Γ. Denote by $a(i, j)$ the number modulo 2 of K-bad edges in $\overline{b_i} \cap \overline{w_j}$. Denote by $u(K)$ the rank of the $m-by-n$ matrix $(a(i, j))$.

THEOREM. *If K is a diagram of a link L, then*

$$(11) \qquad c(K) + r(K) - 1 \geqslant \operatorname{span}(L) + u(K) .$$

COROLLARY. *If K is a diagram of an unsplittable link L, then*

$$c(K) \geqslant \operatorname{span}(L) + u(K) .$$

Of course, if K is a weakly alternating diagram, then $u(K) = 0$.

The inequalities of the Theorem and of the Corollary may be strict. For example, if we take the diagram $K = 8_{19}$ in Rolfsen's book, then span $(8_{19}) = 5$ and $u(K) = 2$, so that the inequality (11) amounts to $8 > 7$. Unfortunately, even in the case where (11) is an equality, it does not mean that K is a minimal diagram of L, since $u(K)$ depends on K and is not an invariant of L.

The proof of the Theorem goes along the same lines as the proof of Theorem 1 of § 1. Indeed the proof of Lemma 1 of § 2 shows in fact that $|S| + |\check{S}| \leqslant c + 2r - R$, where R is the rank of the intersection form (5). For the state A, it is easy to show that $R = 2u(K)$, and this gives the desired result.

REFERENCES

[1] Crowell, R. H. Nonalternating links. *Ill. J. Math. 3* (1959), 101-120.
[2] Gordon, C. McA and R. A. Litherland. On the signature of a link. *Invent. math. 47* (1978), 53-69.
[3] de la Harpe, P., M. Kervaire and C. Weber. On the Jones polynomial. *L'Enseignement math. 32* (1986), 271-335.
[4] Jones, V. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. 12* (1985), 103-111.
[5] Kauffman, L. State models and the Jones polynomial. Preprint, 1986.
[6] Murasugi, K. Jones polynomials and classical conjectures in knot theory. *Topology 26* (1987), 187-194.
[7] Rolfsen, D. *Knots and links.* Publish or Perish 1976.
[8] Tait, P. G. On Knots I, II, III. *Scientific papers, Vol. 1* (1898), 273-347.
[9] Thistlethwaite, M. B. Kauffman's polynomial and alternating links. Preprint, 1986.

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