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**Artikel:** SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS  
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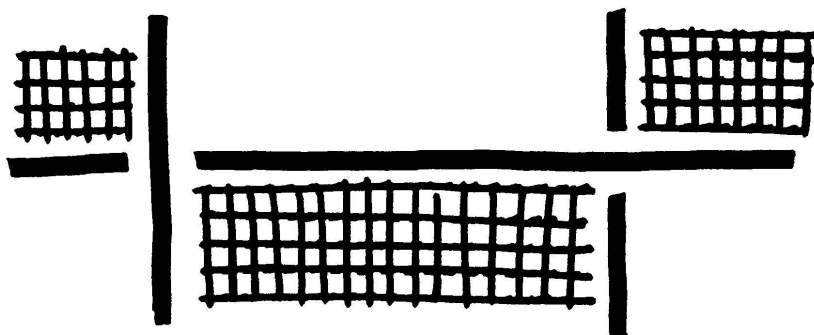


FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram  $K$  would not be reduced:

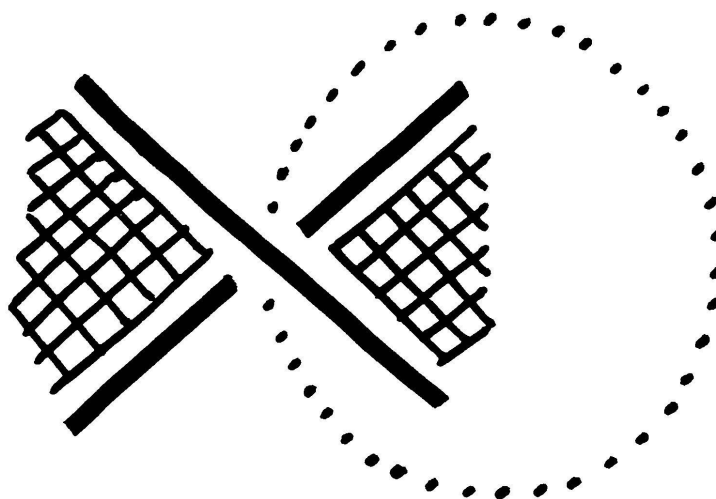


FIGURE 20

It is evident that  $A$  is equal to the number of unshaded regions. Let a state  $S^2$  be obtained from  $A$  by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore  $|S^2| = |A| - 1$ . In view of the arguments given in the proof of part (i) of the Theorem, this implies that  $D_S < D_A$  for any state  $S$  of  $K$ . This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

### § 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link  $L$  in terms of a (not necessarily orientable) surface  $V$  bounded by  $L$  (see [2]). One defines a bilinear form

$$Q = Q_V: H_1(V; \mathbb{Z}) \times H_1(V; \mathbb{Z}) \rightarrow \mathbb{Z}$$

as follows. Let  $\alpha, \beta \in H_1(V; \mathbb{Z})$  be represented by loops  $a, b$  in  $V$ . Let us double all points of  $a$  and push them in  $S^3 - V$  along both normal directions to  $V$ , at the same small distance. We obtain an oriented closed 1-manifold  $\tilde{a} \in S^3 - V$ ; the following picture shows the local situation. The natural projection  $\tilde{a} \rightarrow a$  is of course a 2-sheeted covering.

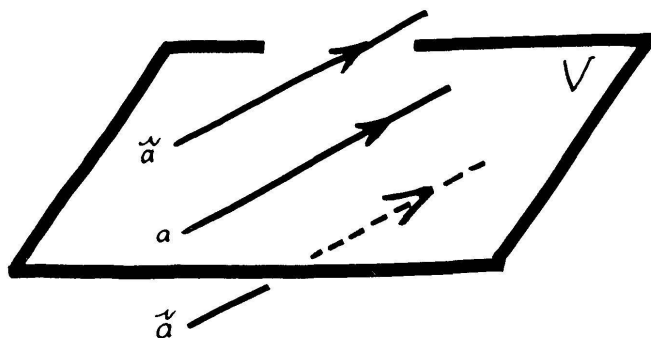


FIGURE 21

Denote by  $Q(\alpha, \beta)$  the linking coefficient  $Lk(\tilde{a}, b)$  of  $\tilde{a}$  and  $b$ . It turns out that  $Q$  is a well defined symmetric bilinear form. Let  $L^V$  be a parallel copy of  $L$  in  $S^3 - V$ . Define

$$\sigma(L) = \text{sign}(Q) - \frac{1}{2} Lk(L, L^V).$$

Here  $\text{sign}(Q)$  denotes the signature of the symmetric bilinear form obtained by factorizing out the annihilator of  $Q$ . According to [2],  $\sigma(L)$  does not depend on the choice of the spanning surface  $V$ . In case  $V$  is orientable,  $Lk(L, L^V) = 0$  and we get the classical definition of the signature of  $L$  due to Murasugi.

All diagrams and links being oriented, it is easy to check that the writhe number of a link diagram, the signature of a link, and the number  $d_{\max}(V_L(t)) + d_{\min}(V_L(t))$  are additive with respect to both disjoint unions and connected sums of diagrams. Therefore it is enough to prove Theorem 2 for a diagram  $K$  which is connected, prime, alternating and reduced.

Let  $c_+$  and  $c_-$  denote the numbers of positive and negative crossing points of such a  $K$ .

CLAIM (Murasugi). *One has  $\sigma(L) = |A| - 1 - c_+$ .*

This claim implies Theorem 2. Indeed, formulas (8), (9) and (6) show that

$$\begin{aligned} & d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K) \\ &= -w(K)/2 + D_A + d_B = -w(K)/2 + (|A| - |B|)/2. \end{aligned}$$

Substituting in the last expression

$$\begin{aligned} w(K) &= c_+ - c_- \\ |B| &= c + 2 - |A| \\ c &= c_+ + c_- \end{aligned}$$

we obtain

$$\begin{aligned} & d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K) \\ &= |A| - 1 - c_+ = \sigma(L). \end{aligned}$$

This implies Theorem 2.

*Proof of the Claim.* There is a spanning surface  $V$  of  $L$  associated with the diagram  $K$ . It is built up from shaded regions of  $S^2 - K$  (see § 4) and small bands connecting these regions which enter one crossing point. In a neighbourhood of a crossing point,  $V$  looks like this:

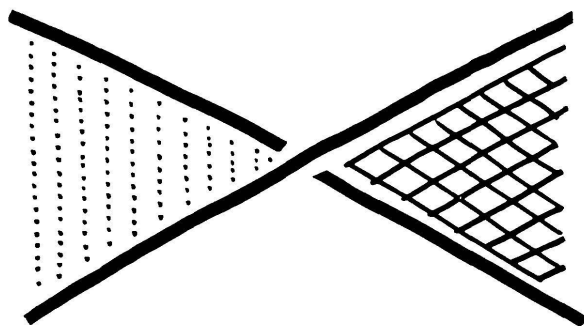


FIGURE 22

We shall prove the claim by using this surface  $V$ .

We prove first that the number  $-\frac{1}{2}Lk(L, L^V)$  is equal to  $-c_+$ . We may assume that the push-off  $L^V$  of  $L$  in  $S^3 - V$  lies in the unshaded regions of  $R^2$  except in a neighbourhood of the crossing points. The following picture shows  $L^V$  near a crossing point (the orientations of  $L$  and  $L^V$  are not shown).

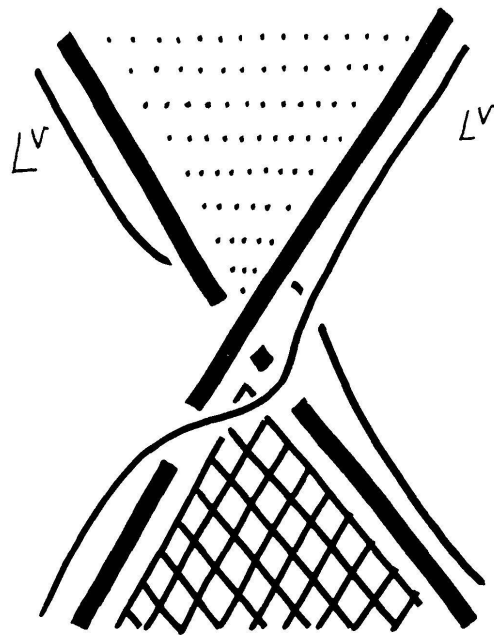


FIGURE 23

We compute  $Lk(L, L^V)$ , by counting the algebraic number of times  $L^V$  passes under  $L$ . It is easy to check that each crossing point of  $L$  contributes with a 2 if it is positive and with a 0 if it is negative. Thus  $Lk(L, L^V) = 2c_+$ .

Now, we prove that  $\text{sign}(Q_V) = |A| - 1$ . The surface  $V$  retracts by deformation onto the complement on the unshaded regions in  $S^2$ . As the diagram is alternating, the number of unshaded regions is  $|A|$ , so that  $b_1(V) = |A| - 1$ . Thus we have to prove that the form  $Q_V$  is positive definite.

Let  $\alpha \in H_1(V; \mathbb{Z})$  and let  $a$  be an oriented closed 1-manifold (possibly non connected) in  $V$  which represents  $\alpha$ . Thus  $Q(\alpha, \alpha) = Lk(\tilde{a}, a)$ , where  $\tilde{a}$  is the oriented closed 1-manifold in  $S^3 - V$  obtained from  $a$  by the 2-sheeted blowing up procedure. If a subarc  $x$  of  $a$  lies in a shaded region far from crossing points of  $K$ , then, of the two corresponding subarcs of  $\tilde{a}$ , one lies over  $R^2$  and the other one lies under  $R^2$ . We shall always picture the first (higher) subarc of  $\tilde{a}$  on the right side of  $x$  (looking from above along  $a$ ) and the second (lower) subarc of  $\tilde{a}$  on the left side of  $x$ ; see the following picture.

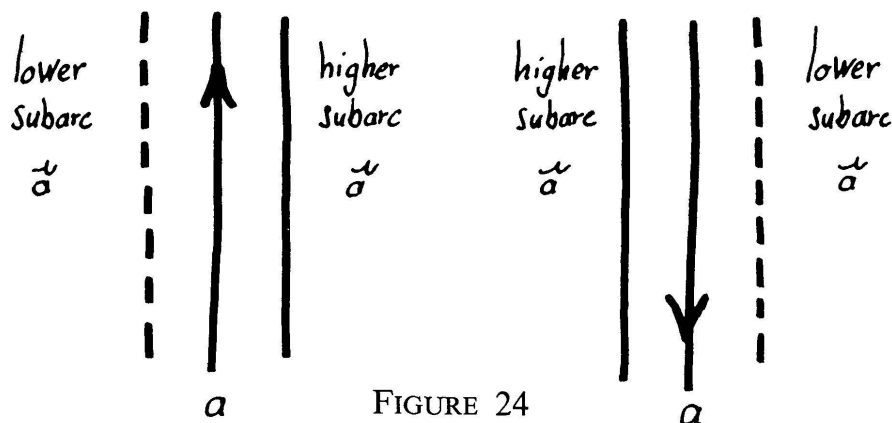


FIGURE 24

Note that the diagram of  $\tilde{a}$  misses the diagram of  $a$  except in a neighborhood of the crossing points. Surgering if necessary  $a$  in  $V$ , we may assume that all components of  $a$  go through any band of  $V$  in one direction. Positions of  $a$  like those in the following picture may easily be removed by surgery.

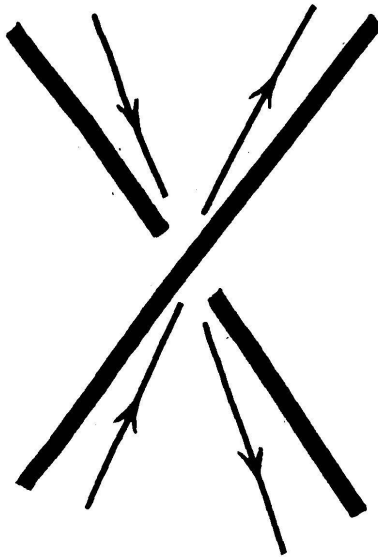


FIGURE 25

For simplicity, consider first a neighbourhood of a crossing point through which  $a$  goes only once:

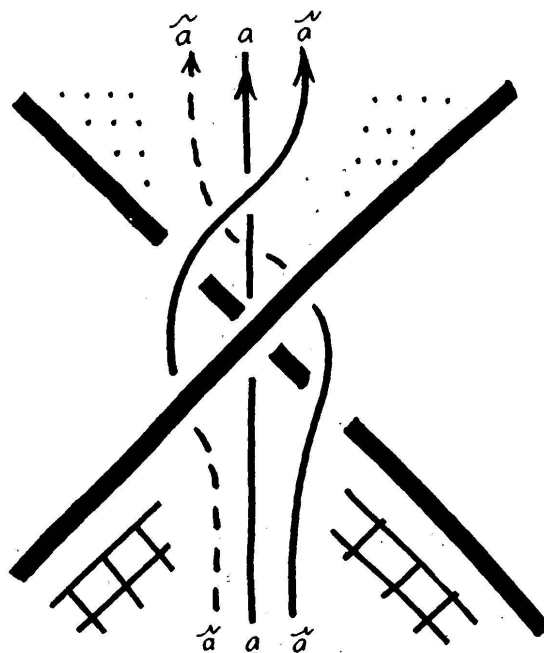


FIGURE 26

It is clear that  $\tilde{a}$  passes under  $a$  in this neighbourhood one time from right to left.

If  $a$  goes through a neighbourhood  $\mathcal{U}$  of a crossing point  $n$  times, then the relative positions of the corresponding  $n$  arcs of  $a$ , say  $x_1, \dots, x_n$ , are represented as follows:

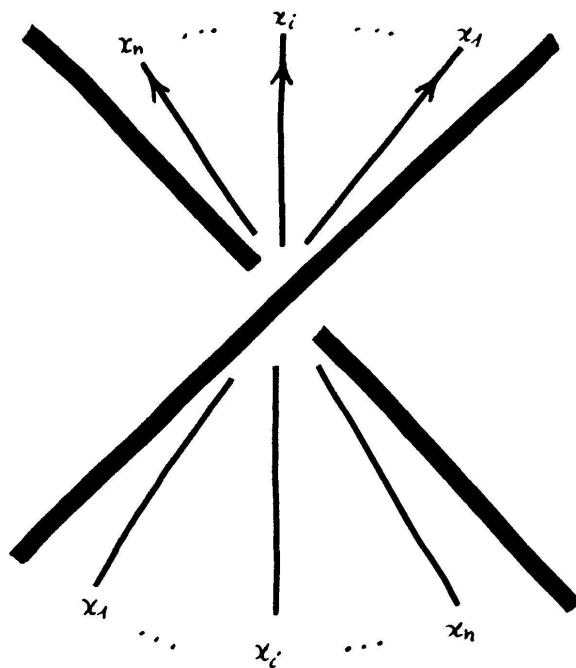


FIGURE 27

In the next picture, we show the two arcs of  $\tilde{a}$  which correspond to  $x_i$ :

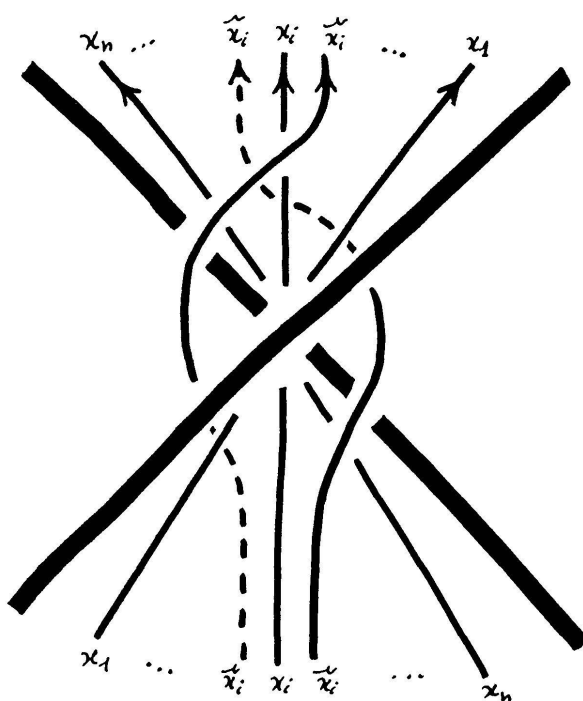


FIGURE 28

It is clear that these two arcs of  $\tilde{a}$  pass  $2i - 1$  times from right to left under  $a$ . Thus the contribution of the neighbourhood  $\mathcal{U}$  to  $Q(\alpha, \alpha)$  is given by

$$\sum_{i=1}^n (2i-1) = -n + 2 \sum_{i=1}^n i = n^2.$$

This shows that  $Q(\alpha, \alpha) > 0$  if  $a$  crosses at least one band of  $V$ . If not, then  $\alpha = 0$ .

Thus  $Q$  is positive definite. This completes the proof of Theorem 2.

#### APPENDIX: AN IMPROVEMENT OF THE INEQUALITY OF THEOREM 1

Though the inequality

$$(10) \quad c(K) + r(K) - 1 \geq \text{span}(L)$$

of Theorem 1 becomes an equality for weakly alternating diagrams, it may be sharpened a little for other cases. Let  $K$  be a link diagram in  $R^2$  and let  $\Gamma \subset R^2$  be the associated link projection. For  $P \in S^2 - \Gamma$  (where  $S^2 = R^2 \cup \{\infty\}$ ), let  $i(P)$  be the intersection number modulo 2 of  $\Gamma$  with a generic 1-chain connecting  $P$  to  $\infty$ . Shade the regions of  $S^2 - \Gamma$  for which  $i \equiv 1 \pmod{2}$ , so that  $S^2$  is painted like a chessboard. Let  $b_1, \dots, b_m$  be the shaded regions of  $S^2 - \Gamma$  and let  $w_1, \dots, w_n$  be the unshaded regions of  $S^2 - \Gamma$ .

An edge  $e$  of  $\Gamma$  is called *K-good* either if  $e$  is a loop or if one of the end points of  $e$  corresponds to an overcrossing point of  $K$  and the other end point of  $e$  corresponds to an undercrossing point of  $K$ . An edge of  $\Gamma$  which is not *K-good* is called *K-bad*. For any  $i \in \{1, \dots, m\}$  and for any  $j \in \{1, \dots, n\}$ , it is clear that the set  $\overline{b_i} \cap \overline{w_j}$  consists of several edges and double points of  $\Gamma$ . Denote by  $a(i, j)$  the number modulo 2 of *K-bad* edges in  $\overline{b_i} \cap \overline{w_j}$ . Denote by  $u(K)$  the rank of the  $m \times n$  matrix  $(a(i, j))$ .

**THEOREM.** *If  $K$  is a diagram of a link  $L$ , then*

$$(11) \quad c(K) + r(K) - 1 \geq \text{span}(L) + u(K).$$

**COROLLARY.** *If  $K$  is a diagram of an unsplittable link  $L$ , then*

$$c(K) \geq \text{span}(L) + u(K).$$

Of course, if  $K$  is a weakly alternating diagram, then  $u(K) = 0$ .