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SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS
Turaev, V. G.
§5. Proof of Theorem 2
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FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram K would not be reduced:



FIGURE 20

It is evident that A is equal to the number of unshaded regions. Let a state S^2 be obtained from A by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore $|S^2| = |A| - 1$. In view of the arguments given in the proof of part (i) of the Theorem, this implies that $D_S < D_A$ for any state S of K. This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

§ 5. Proof of Theorem 2

Let me first recall the definition of the signature of an oriented link L in terms of a (not necessarily orientable) surface V bounded by L (see [2]). One defines a bilinear form

$$Q = Q_V \colon H_1(V; Z) \times H_1(V; Z) \to Z$$

as follows. Let α , $\beta \in H_1(V; Z)$ be represented by loops a, b in V. Let us double all points of a and push them in $S^3 - V$ along both normal directions to V, at the same small distance. We obtain an oriented closed 1-manifold $\tilde{a} \in S^3 - V$; the following picture shows the local situation. The natural projection $\tilde{a} \to a$ is of course a 2-sheeted covering.



FIGURE 21

Denote by $Q(\alpha, \beta)$ the linking coefficient $Lk(\tilde{a}, b)$ of \tilde{a} and b. It turns out that Q is a well defined symmetric bilinear form. Let L^{V} be a parallel copy of L in $S^{3} - V$. Define

$$\sigma(L) = \operatorname{sign}(Q) - \frac{1}{2} Lk(L, L^{V}).$$

Here sign (Q) denotes the signature of the symmetric bilinear form obtained by factorizing out the annihilator of Q. According to [2], $\sigma(L)$ does not depend on the choice of the spanning surface V. In case V is orientable, $Lk(L, L^{V}) = 0$ and we get the classical definition of the signature of Ldue to Murasugi.

All diagrams and links being oriented, it is easy to check that the writhe number of a link diagram, the signature of a link, and the number $d_{\max}(V_L(t)) + d_{\min}(V_L(t))$ are additive with respect to both disjoint unions and connected sums of diagrams. Therefore it is enough to prove Theorem 2 for a diagram K which is connected, prime, alternating and reduced.

Let c_+ and c_- denote the numbers of positive and negative crossing points of such a K.

CLAIM (Murasugi). One has $\sigma(L) = |A| - 1 - c_+$.

This claim implies Theorem 2. Indeed, formulas (8), (9) and (6) show that

$$d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K)$$

= $-w(K)/2 + D_A + d_B = -w(K)/2 + (|A| - |B|)/2$.

Substituting in the last expression

$$w(K) = c_{+} - c_{-}$$

| B | = c + 2 - | A |
$$c = c_{+} + c_{-}$$

we obtain

$$d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K) = |A| - 1 - c_+ = \sigma(L).$$

This implies Theorem 2.

Proof of the Claim. There is a spanning surface V of L associated with the diagram K. It is built up from shaded regions of $S^2 - K$ (see § 4) and small bands connecting these regions which enter one crossing point. In a neighbourhood of a crossing point, V looks like this:



FIGURE 22

We shall prove the claim by using this surface V.

We prove first that the number $-\frac{1}{2}Lk(L, L^{V})$ is equal to $-c_{+}$. We may assume that the push-off L^{V} of L in $S^{3} - V$ lies in the unshaded regions of R^{2} except in a neighbourhood of the crossing points. The following picture shows L^{V} near a crossing point (the orientations of L and L^{V} are not shown).



FIGURE 23

We compute $Lk(L, L^{V})$, by counting the algebraic number of times L^{V} passes under L. It is easy to check that each crossing point of L contributes with a 2 if it is positive and with a 0 if it is negative. Thus $Lk(L, L^{V}) = 2c_{+}$.

Now, we prove that sign $(Q_V) = |A| - 1$. The surface V retracts by deformation onto the complement on the unshaded regions in S^2 . As the diagram is alternating, the number of unshaded regions is |A|, so that $b_1(V) = |A| - 1$. Thus we have to prove that the form Q_V is positive definite.

Let $\alpha \in H_1(V; Z)$ and let *a* be an oriented closed 1-manifold (possibly non connected) in *V* which represents α . Thus $Q(\alpha, \alpha) = Lk(\tilde{a}, a)$, where \tilde{a} is the oriented closed 1-manifold in $S^3 - V$ obtained from *a* by the 2-sheeted blowing up procedure. If a subarc *x* of *a* lies in a shaded region far from crossing points of *K*, then, of the two corresponding subarcs of \tilde{a} , one lies over R^2 and the other one lies under R^2 . We shall always picture the first (higher) subarc of \tilde{a} on the right side of *x* (looking from above along *a*) and the second (lower) subarc of \tilde{a} on the left side of *x*; see the following picture.



Note that the diagram of \tilde{a} misses the diagram of a except in a neighborhood of the crossing points. Surgering if necessary a in V, we may assume that all components of a go through any band of V in one direction. Positions of a like those in the following picture may easily be removed by surgery.



FIGURE 25

For simplicity, consider first a neighbourhood of a crossing point through which a goes only once:



FIGURE 26

It is clear that \tilde{a} passes under a in this neighbourhood one time from right to left.

If a goes through a neighbourhood \mathcal{U} of a crossing point n times, then the relative positions of the corresponding n arcs of a, say $x_1, ..., x_n$, are represented as follows:



FIGURE 27

In the next picture, we show the two arcs of \tilde{a} which correspond to x_i :



It is clear that these two arcs of \tilde{a} pass 2i - 1 times from right to left under a. Thus the contribution of the neighbourhood \mathcal{U} to $Q(\alpha, \alpha)$ is given by

$$\sum_{i=1}^{n} (2i-1) = -n + 2 \sum_{i=1}^{n} i = n^{2}.$$

This shows that $Q(\alpha, \alpha) > 0$ if a crosses at least one band of V. If not, then $\alpha = 0$.

Thus Q is positive definite. This completes the proof of Theorem 2.

APPENDIX: AN IMPROVEMENT OF THE INEQUALITY OF THEOREM 1

Though the inequality

(10)
$$c(K) + r(K) - 1 \ge \operatorname{span}(L)$$

of Theorem 1 becomes an equality for weakly alternating diagrams, it may be sharpened a little for other cases. Let K be a link diagram in R^2 and let $\Gamma \subset R^2$ be the associated link projection. For $P \in S^2 - \Gamma$ (where $S^2 = R^2 \cup \{\infty\}$), let i(P) be the intersection number modulo 2 of Γ with a generic 1-chain connecting P to ∞ . Shade the regions of $S^2 - \Gamma$ for which $i \equiv 1 \pmod{2}$, so that S^2 is painted like a chessboard. Let $b_1, ..., b_m$ be the shaded regions of $S^2 - \Gamma$ and let $w_1, ..., w_n$ be the unshaded regions of $S^2 - \Gamma$.

An edge e of Γ is called *K*-good either if e is a loop or if one of the end points of e corresponds to an overcrossing point of K and the other end point of e corresponds to an undercrossing point of K. An edge of Γ which is not K-good is called K-bad. For any $i \in \{1, ..., m\}$ and for any $j \in \{1, ..., n\}$, it is clear that the set $\overline{b_i} \cap \overline{w_j}$ consists of several edges and double points of Γ . Denote by a(i, j) the number modulo 2 of K-bad edges in $\overline{b_i} \cap \overline{w_j}$. Denote by u(K) the rank of the m-by-n matrix (a(i, j)).

THEOREM. If K is a diagram of a link L, then

(11)
$$c(K) + r(K) - 1 \ge \operatorname{span}(L) + u(K).$$

COROLLARY. If K is a diagram of an unsplittable link L, then $c(K) \ge \operatorname{span}(L) + u(K)$.

Of course, if K is a weakly alternating diagram, then u(K) = 0.