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ON ALTERNATING LINKS
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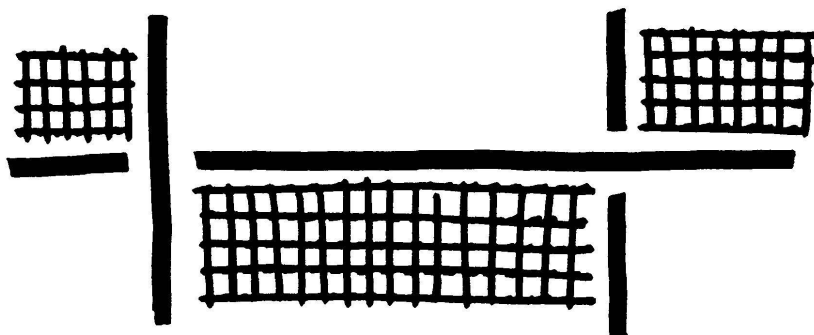


FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram K would not be reduced:

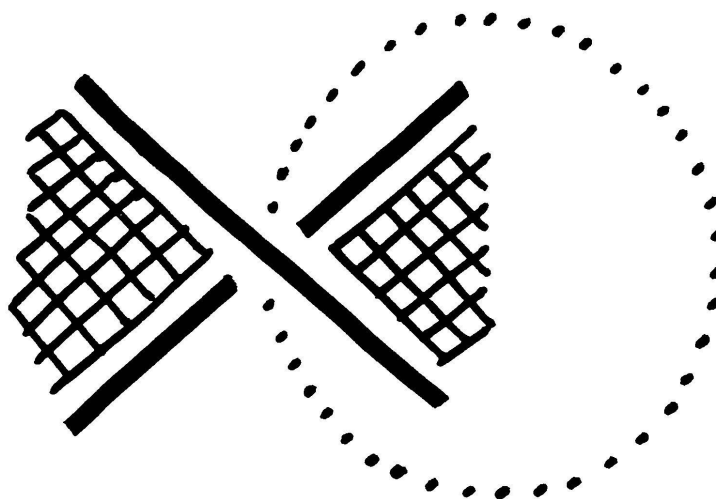


FIGURE 20

It is evident that A is equal to the number of unshaded regions. Let a state S^2 be obtained from A by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore $|S^2| = |A| - 1$. In view of the arguments given in the proof of part (i) of the Theorem, this implies that $D_S < D_A$ for any state S of K . This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

§ 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link L in terms of a (not necessarily orientable) surface V bounded by L (see [2]). One defines a bilinear form

$$Q = Q_V: H_1(V; \mathbb{Z}) \times H_1(V; \mathbb{Z}) \rightarrow \mathbb{Z}$$

as follows. Let $\alpha, \beta \in H_1(V; \mathbb{Z})$ be represented by loops a, b in V . Let us double all points of a and push them in $S^3 - V$ along both normal directions to V , at the same small distance. We obtain an oriented closed 1-manifold $\tilde{a} \in S^3 - V$; the following picture shows the local situation. The natural projection $\tilde{a} \rightarrow a$ is of course a 2-sheeted covering.

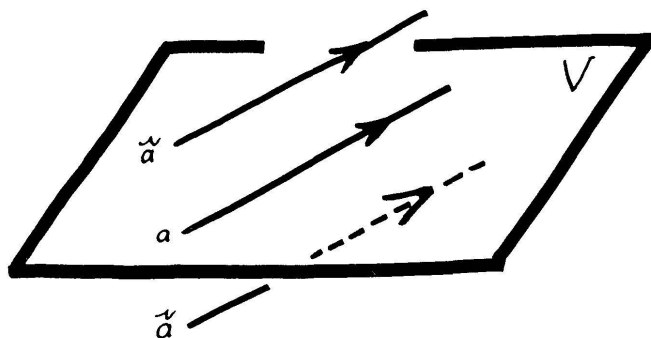


FIGURE 21

Denote by $Q(\alpha, \beta)$ the linking coefficient $Lk(\tilde{a}, b)$ of \tilde{a} and b . It turns out that Q is a well defined symmetric bilinear form. Let L^V be a parallel copy of L in $S^3 - V$. Define

$$\sigma(L) = \text{sign}(Q) - \frac{1}{2} Lk(L, L^V).$$

Here $\text{sign}(Q)$ denotes the signature of the symmetric bilinear form obtained by factorizing out the annihilator of Q . According to [2], $\sigma(L)$ does not depend on the choice of the spanning surface V . In case V is orientable, $Lk(L, L^V) = 0$ and we get the classical definition of the signature of L due to Murasugi.

All diagrams and links being oriented, it is easy to check that the writhe number of a link diagram, the signature of a link, and the number $d_{\max}(V_L(t)) + d_{\min}(V_L(t))$ are additive with respect to both disjoint unions and connected sums of diagrams. Therefore it is enough to prove Theorem 2 for a diagram K which is connected, prime, alternating and reduced.

Let c_+ and c_- denote the numbers of positive and negative crossing points of such a K .

CLAIM (Murasugi). *One has $\sigma(L) = |A| - 1 - c_+$.*

This claim implies Theorem 2. Indeed, formulas (8), (9) and (6) show that

$$\begin{aligned}
 & d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K) \\
 &= -w(K)/2 + D_A + d_B = -w(K)/2 + (|A| - |B|)/2.
 \end{aligned}$$

Substituting in the last expression

$$\begin{aligned}
 w(K) &= c_+ - c_- \\
 |B| &= c + 2 - |A| \\
 c &= c_+ + c_-
 \end{aligned}$$

we obtain

$$\begin{aligned}
 & d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K) \\
 &= |A| - 1 - c_+ = \sigma(L).
 \end{aligned}$$

This implies Theorem 2.

Proof of the Claim. There is a spanning surface V of L associated with the diagram K . It is built up from shaded regions of $S^2 - K$ (see § 4) and small bands connecting these regions which enter one crossing point. In a neighbourhood of a crossing point, V looks like this:

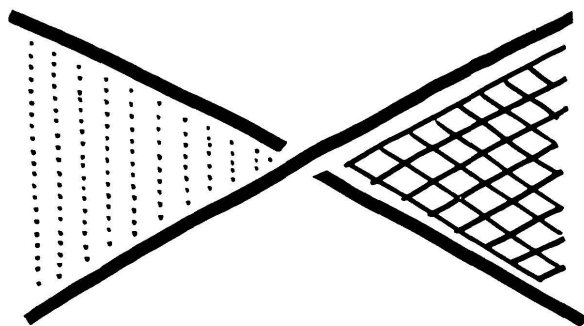


FIGURE 22

We shall prove the claim by using this surface V .

We prove first that the number $-\frac{1}{2}Lk(L, L^V)$ is equal to $-c_+$. We may assume that the push-off L^V of L in $S^3 - V$ lies in the unshaded regions of R^2 except in a neighbourhood of the crossing points. The following picture shows L^V near a crossing point (the orientations of L and L^V are not shown).

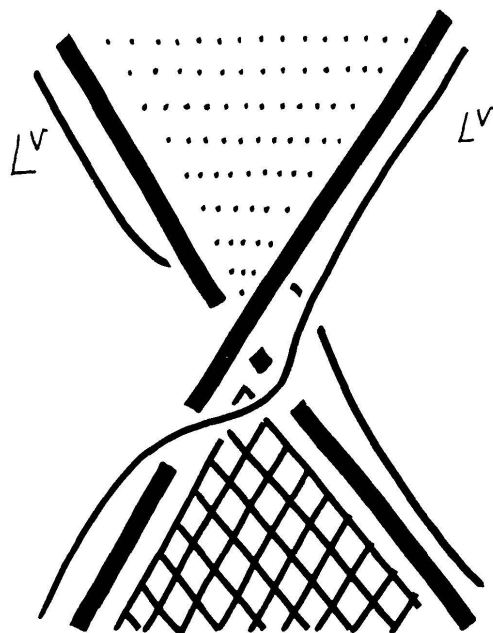


FIGURE 23

We compute $Lk(L, L^V)$, by counting the algebraic number of times L^V passes under L . It is easy to check that each crossing point of L contributes with a 2 if it is positive and with a 0 if it is negative. Thus $Lk(L, L^V) = 2c_+$.

Now, we prove that $\text{sign}(Q_V) = |A| - 1$. The surface V retracts by deformation onto the complement on the unshaded regions in S^2 . As the diagram is alternating, the number of unshaded regions is $|A|$, so that $b_1(V) = |A| - 1$. Thus we have to prove that the form Q_V is positive definite.

Let $\alpha \in H_1(V; \mathbb{Z})$ and let a be an oriented closed 1-manifold (possibly non connected) in V which represents α . Thus $Q(\alpha, \alpha) = Lk(\tilde{a}, a)$, where \tilde{a} is the oriented closed 1-manifold in $S^3 - V$ obtained from a by the 2-sheeted blowing up procedure. If a subarc x of a lies in a shaded region far from crossing points of K , then, of the two corresponding subarcs of \tilde{a} , one lies over R^2 and the other one lies under R^2 . We shall always picture the first (higher) subarc of \tilde{a} on the right side of x (looking from above along a) and the second (lower) subarc of \tilde{a} on the left side of x ; see the following picture.

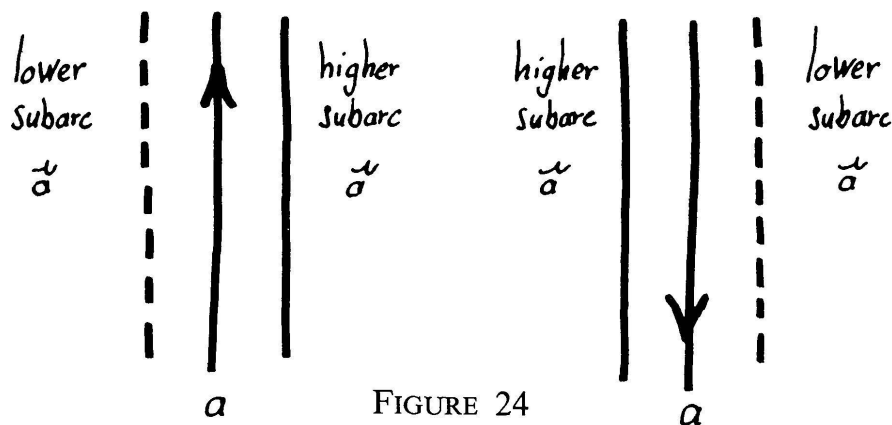


FIGURE 24

Note that the diagram of \tilde{a} misses the diagram of a except in a neighborhood of the crossing points. Surgering if necessary a in V , we may assume that all components of a go through any band of V in one direction. Positions of a like those in the following picture may easily be removed by surgery.

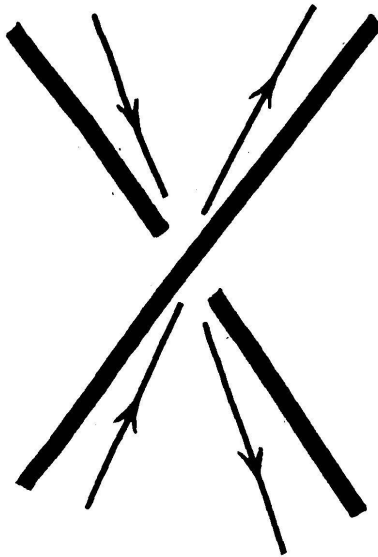


FIGURE 25

For simplicity, consider first a neighbourhood of a crossing point through which a goes only once:

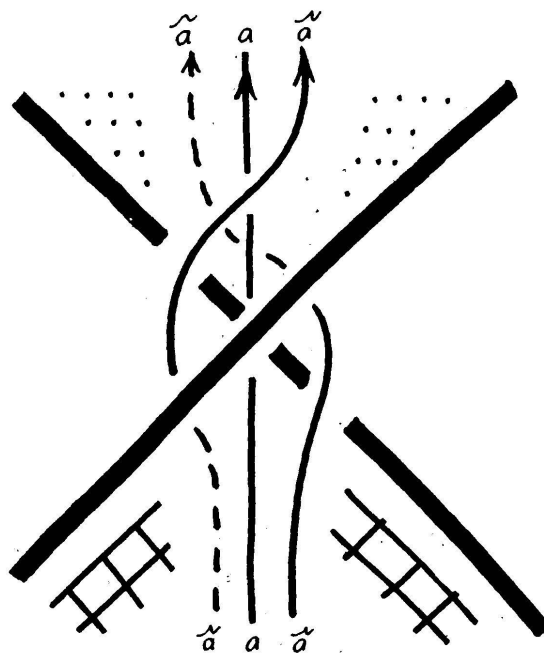


FIGURE 26

It is clear that \tilde{a} passes under a in this neighbourhood one time from right to left.

If a goes through a neighbourhood \mathcal{U} of a crossing point n times, then the relative positions of the corresponding n arcs of a , say x_1, \dots, x_n , are represented as follows:

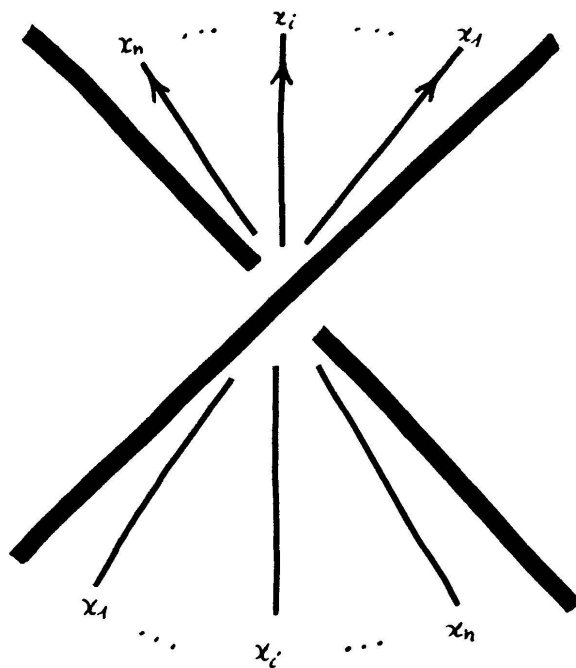


FIGURE 27

In the next picture, we show the two arcs of \tilde{a} which correspond to x_i :

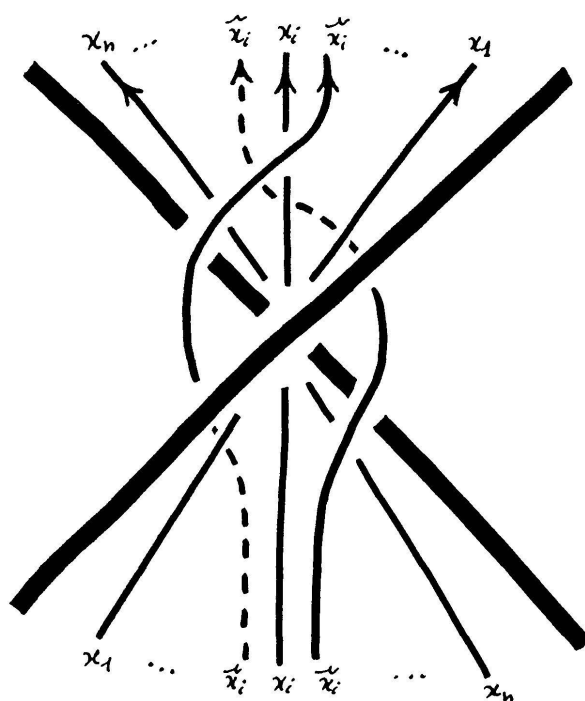


FIGURE 28

It is clear that these two arcs of \tilde{a} pass $2i - 1$ times from right to left under a . Thus the contribution of the neighbourhood \mathcal{U} to $Q(\alpha, \alpha)$ is given by

$$\sum_{i=1}^n (2i-1) = -n + 2 \sum_{i=1}^n i = n^2.$$

This shows that $Q(\alpha, \alpha) > 0$ if a crosses at least one band of V . If not, then $\alpha = 0$.

Thus Q is positive definite. This completes the proof of Theorem 2.

APPENDIX: AN IMPROVEMENT OF THE INEQUALITY OF THEOREM 1

Though the inequality

$$(10) \quad c(K) + r(K) - 1 \geq \text{span}(L)$$

of Theorem 1 becomes an equality for weakly alternating diagrams, it may be sharpened a little for other cases. Let K be a link diagram in R^2 and let $\Gamma \subset R^2$ be the associated link projection. For $P \in S^2 - \Gamma$ (where $S^2 = R^2 \cup \{\infty\}$), let $i(P)$ be the intersection number modulo 2 of Γ with a generic 1-chain connecting P to ∞ . Shade the regions of $S^2 - \Gamma$ for which $i \equiv 1 \pmod{2}$, so that S^2 is painted like a chessboard. Let b_1, \dots, b_m be the shaded regions of $S^2 - \Gamma$ and let w_1, \dots, w_n be the unshaded regions of $S^2 - \Gamma$.

An edge e of Γ is called *K-good* either if e is a loop or if one of the end points of e corresponds to an overcrossing point of K and the other end point of e corresponds to an undercrossing point of K . An edge of Γ which is not *K-good* is called *K-bad*. For any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, n\}$, it is clear that the set $\overline{b_i} \cap \overline{w_j}$ consists of several edges and double points of Γ . Denote by $a(i, j)$ the number modulo 2 of *K-bad* edges in $\overline{b_i} \cap \overline{w_j}$. Denote by $u(K)$ the rank of the $m \times n$ matrix $(a(i, j))$.

THEOREM. *If K is a diagram of a link L , then*

$$(11) \quad c(K) + r(K) - 1 \geq \text{span}(L) + u(K).$$

COROLLARY. *If K is a diagram of an unsplittable link L , then*

$$c(K) \geq \text{span}(L) + u(K).$$

Of course, if K is a weakly alternating diagram, then $u(K) = 0$.