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## § 3. KAUFFMAN'S STATE MODEL FOR THE JONES POLYNOMIAL

Let  $K$  be a link diagram. By a state or a marker of  $K$ , we mean respectively a state or a marker of the corresponding link projection in  $R^2$  (which is obtained from  $K$  by forgetting the overcrossing-undercrossing data). The markers of  $K$  are divided into two classes — positive and negative. By definition, if the over-line is rotated counterclockwise around the double point, then the first marker it meets is the positive one and the second one is negative:

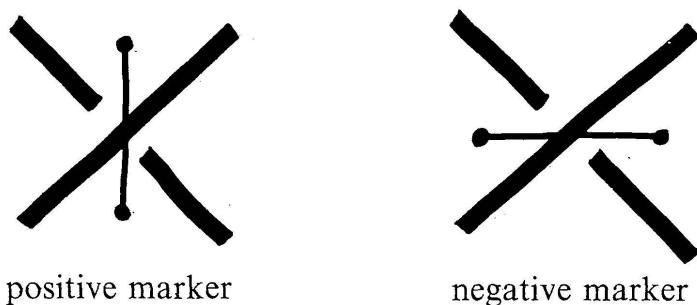


FIGURE 17

Let the diagram  $K$  be oriented. Consider the polynomial

$$V_K(t) = (-t)^{-3w(K)/4} \sum t^{(as - bs)/4} (-t^{1/2} - t^{-1/2})^{|S| - 1}$$

where  $w(K)$  is the writhe number of  $K$ . The summation is over all the states  $S$  of  $K$ ; the number of positive [respectively negative] markers of the state  $S$  is denoted by  $a_s$  [respectively  $b_s$ ], and the number  $|S|$  is defined in § 2.

It is shown in [5] that the polynomial  $V_K(t)$  is equal to the Jones polynomial of the oriented link presented by  $K$  (see also [3]).

## § 4. PROOF OF THEOREM 1

Orient the diagram  $K$  and denote the corresponding oriented link by  $L$ . Denote by  $A$  the state of  $K$  in which all markers are positive, and by  $B = \bar{A}$  the dual state in which all markers are negative. For any state  $S$  of  $K$ , denote by  $D_S$  and  $d_S$  respectively the maximal and minimal degrees in  $t$  in the expression

$$t^{(as - bs)/4} (-t^{1/2} - t^{-1/2})^{|S| - 1}$$

(see § 3), namely

$$D_S = (a_S - b_S + 2|S| - 2)/4$$

$$d_S = (a_S - b_S - 2|S| + 2)/4.$$

In particular

$$(6) \quad \begin{aligned} D_A &= (c + 2|A| - 2)/4 \\ d_B &= (-c - 2|B| + 2)/4. \end{aligned}$$

*Proof of (i).* If a state  $S^2$  is obtained from a state  $S$  by replacing one positive marker by a negative one (at some crossing point), then  $a_{S^2} = a_S - 1$ ,  $b_{S^2} = b_S + 1$  and  $|S^2| \leq |S| + 1$ . Thus

$$D_{S^2} - D_S = -\frac{1}{2} + (|S^2| - |S|)/2 \leq 0$$

so that  $D_{S^2} \leq D_S$ . This implies that  $D_S \leq D_A$  for any state  $S$  of  $K$ . Therefore

$$d_{\max}(V_L(t)) \leq -\frac{3}{4}w(K) + D_A$$

$$d_{\min}(V_L(t)) \geq -\frac{3}{4}w(K) + d_B.$$

Thus in view of equalities (6) and of Lemma 1 of § 2, one has

$$(7) \quad \begin{aligned} \text{span}(L) &\leq D_A - d_B = (c + |A| + |B| - 2)/2 \\ &\leq (2c + 2r - 2)/2 = c + r - 1. \end{aligned}$$

□

*Proof of (ii).* Let  $K_1, \dots, K_r$  be the unsplittable components of  $K$ , with  $r = r(K)$ . Denote by  $L_i$  the oriented link represented by  $K_i$ . It follows from part (i) of the Theorem and from formula (1) that

$$c(K) = \sum_{i=1}^r c(K_i) \geq \sum_{i=1}^r \text{span}(L_i) = \text{span}(L) - (r-1).$$

Thus the equality  $c(K) + r - 1 = \text{span}(L)$  holds if and only if  $c(K_i) = \text{span}(L_i)$  for each  $i$ . Therefore, to prove (ii), it suffices to consider the unsplittable case  $r = 1$ .

It is evident that the numbers  $c(K)$  and  $\text{span}(L)$  are both additive under connected sum of diagrams. Therefore it is enough to prove the following assertion (\*).

(\*) { For a prime unsplittable diagram  $K$  of an oriented link  $L$ , the equality  $c(K) = \text{span}(L)$  holds if and only if  $K$  is a reduced and alternating diagram.

In (\*), note that, formally, the link  $L$  is not supposed to be prime or even unsplittable.

Suppose first that  $c(K) = \text{span}(L)$ . Then all inequalities above are in fact equalities. As  $r = 1$ , one has in particular

$$|A| + |B| = c + 2r = c + 2.$$

Lemma 2 of § 2 shows that the state  $A$  is monochrome. This implies that  $K$  is alternating, because of the easy but essential lemma :

LEMMA. *Let  $K$  be an oriented connected link diagram. Then  $K$  is alternating if and only if the state  $A$  is monochrome.*

Moreover the diagram  $K$  is reduced, since all prime diagrams are reduced except the two diagrams

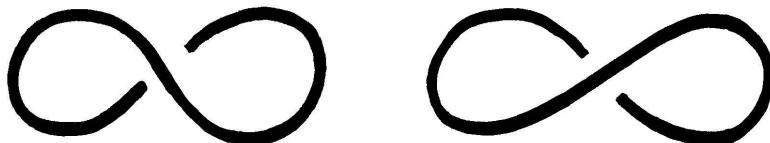


FIGURE 18

which are excluded by the assumption  $c(K) = \text{span}(L)$ .

Suppose conversely that  $K$  is reduced and alternating. The preceding Lemma shows that the state  $A$  is monochrome. According to Lemma 2 of § 2:  $|A| + |B| = c + 2$ . We prove below that

$$(8) \quad d_{\max}(V_L(t)) = -\frac{3}{4}w(K) + D_A$$

$$(9) \quad d_{\min}(V_L(t)) = -\frac{3}{4}w(K) + d_B.$$

Thus the inequalities (7) are in fact equalities, so that  $\text{span}(L) = c + r - 1 = c$ .

By region, we mean hereafter a connected component of  $S^2 - K$ . (Here  $S^2 = R^2 \cup \{\infty\}$ .) Since  $K$  is alternating, each region intersects either markers which are all positive or markers which are all negative. Shade the regions of the first type:

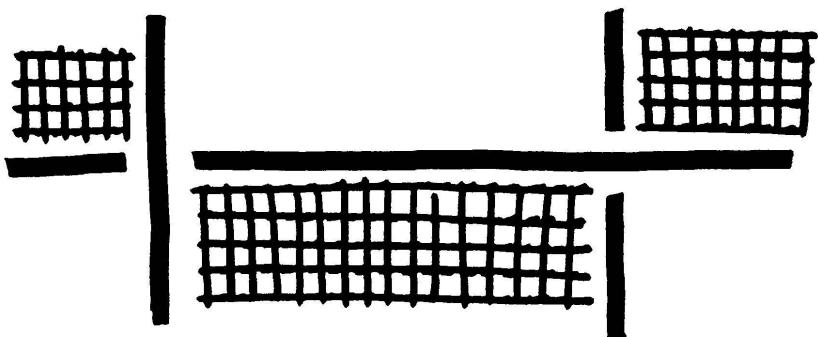


FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram  $K$  would not be reduced:

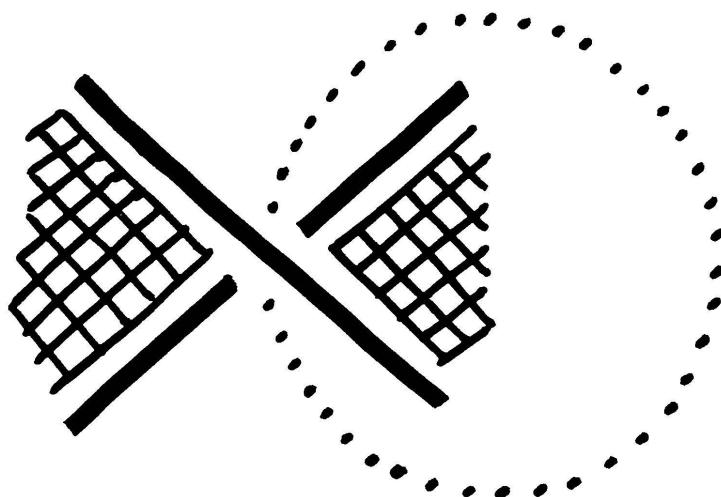


FIGURE 20

It is evident that  $A$  is equal to the number of unshaded regions. Let a state  $S^2$  be obtained from  $A$  by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore  $|S^2| = |A| - 1$ . In view of the arguments given in the proof of part (i) of the Theorem, this implies that  $D_S < D_A$  for any state  $S$  of  $K$ . This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

## § 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link  $L$  in terms of a (not necessarily orientable) surface  $V$  bounded by  $L$  (see [2]). One defines a bilinear form