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Artikel: SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS

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Kapitel: §4. Proof of Theorem 1

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§ 3. Kauffman's state model for the Jones polynomial

Let K be a link diagram. By a state or a marker of K, we mean respectively a state or a marker of the corresponding link projection in R^2 (which is obtained from K by forgetting the overcrossing-undercrossing data). The markers of K are divided into two classes — positive and negative. By definition, if the over-line is rotated counterclockwise around the double point, then the first marker it meets is the positive one and the second one is negative:

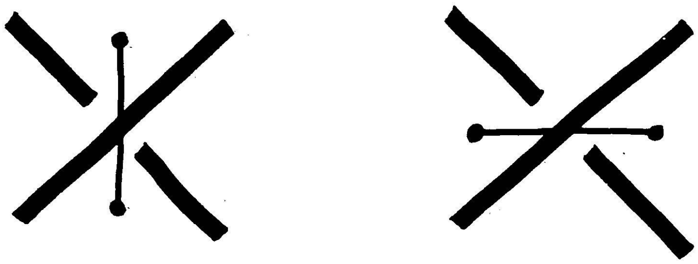

Figure 17

Let the diagram K be oriented. Consider the polynomial

$$V_K(t) = (-t)^{-3w(K)/4} \sum t^{(a_S - b_S)/4}(-t^{1/2} - t^{-1/2})^{|S|-1}$$

where $w(K)$ is the writhe number of K. The summation is over all the states S of K; the number of positive [respectively negative] markers of the state S is denoted by a_S [respectively b_S], and the number $|S|$ is defined in § 2.

It is shown in [5] that the polynomial $V_K(t)$ is equal to the Jones polynomial of the oriented link presented by K (see also [3]).

§ 4. Proof of Theorem 1

Orient the diagram K and denote the corresponding oriented link by L. Denote by A the state of K in which all markers are positive, and by $B = \check{A}$ the dual state in which all markers are negative. For any state S of K, denote by D_S and d_S respectively the maximal and minimal degrees in t in the expression

$$t^{(a_S - b_S)/4}(-t^{1/2} - t^{-1/2})^{|S|-1}$$

(see § 3), namely

$$D_S = (a_S - b_S + 2|S| - 2)/4$$
$$d_S = (a_S - b_S - 2|S| + 2)/4 .$$

In particular

$$(6) \qquad \begin{aligned} D_A &= (c + 2|A| - 2)/4 \\ d_B &= (-c - 2|B| + 2)/4 . \end{aligned}$$

Proof of (i). If a state S^2 is obtained from a state S by replacing one positive marker by a negative one (at some crossing point), then $a_{S^2} = a_S - 1$, $b_{S^2} = b_S + 1$ and $| S^2 | \leqslant | S | + 1$. Thus

$$D_{S^2} - D_S = -\frac{1}{2} + (|S^2| - |S|)/2 \leqslant 0$$

so that $D_{S^2} \leqslant D_S$. This implies that $D_S \leqslant D_A$ for any state S of K. Therefore

$$d_{\max}(V_L(t)) \leqslant -\frac{3}{4} w(K) + D_A$$
$$d_{\min}(V_L(t)) \geqslant -\frac{3}{4} w(K) + d_B .$$

Thus in view of equalities (6) and of Lemma 1 of § 2, one has

$$(7) \qquad \begin{aligned} \operatorname{span}(L) &\leqslant D_A - d_B = (c + |A| + |B| - 2)/2 \\ &\leqslant (2c + 2r - 2)/2 = c + r - 1 . \end{aligned} \qquad \square$$

Proof of (ii). Let $K_1, \dots, K_r$ be the unsplittable components of K, with $r = r(K)$. Denote by L_i the oriented link represented by K_i. It follows from part (i) of the Theorem and from formula (1) that

$$c(K) = \sum_{i=1}^{r} c(K_i) \geqslant \sum_{i=1}^{r} \operatorname{span}(L_i) = \operatorname{span}(L) - (r-1) .$$

Thus the equality $c(K) + r - 1 = \operatorname{span}(L)$ holds if and only if $c(K_i) = \operatorname{span}(L_i)$ for each i. Therefore, to prove (ii), it suffices to consider the unsplittable case $r = 1$.

It is evident that the numbers $c(K)$ and $\operatorname{span}(L)$ are both additive under connected sum of diagrams. Therefore it is enough to prove the following assertion (*).

(*) For a prime unsplittable diagram K of an oriented link L, the equality $c(K) = \operatorname{span}(L)$ holds if and only if K is a reduced and alternating diagram.

In (*), note that, formally, the link L is not supposed to be prime or even unsplittable.

Suppose first that $c(K) = \operatorname{span}(L)$. Then all inequalities above are in fact equalities. As $r = 1$, one has in particular

$$|A| + |B| = c + 2r = c + 2\,.$$

Lemma 2 of § 2 shows that the state A is monochrome. This implies that K is alternating, because of the easy but essential lemma:

LEMMA. *Let K be an oriented connected link diagram. Then K is alternating if and only if the state A is monochrome.*

Moreover the diagram K is reduced, since all prime diagrams are reduced except the two diagrams

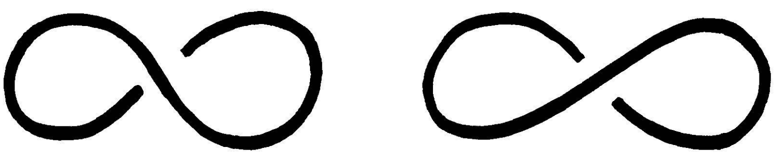

FIGURE 18

which are excluded by the assumption $c(K) = \operatorname{span}(L)$.

Suppose conversely that K is reduced and alternating. The preceeding Lemma shows that the state A is monochrome. According to Lemma 2 of § 2: $|A| + |B| = c + 2$. We prove below that

$$d_{\max}(V_L(t)) = -\frac{3}{4}w(K) + D_A \tag{8}$$

$$d_{\min}(V_L(t)) = -\frac{3}{4}w(K) + d_B\,. \tag{9}$$

Thus the inequalities (7) are in fact equalities, so that $\operatorname{span}(L) = c + r - 1 = c$.

By region, we mean hereafter a connected component of $S^2 - K$. (Here $S^2 = R^2 \cup \{\infty\}$.) Since K is alternating, each region intersects either markers which are all positive or markers which are all negative. Shade the regions of the first type:

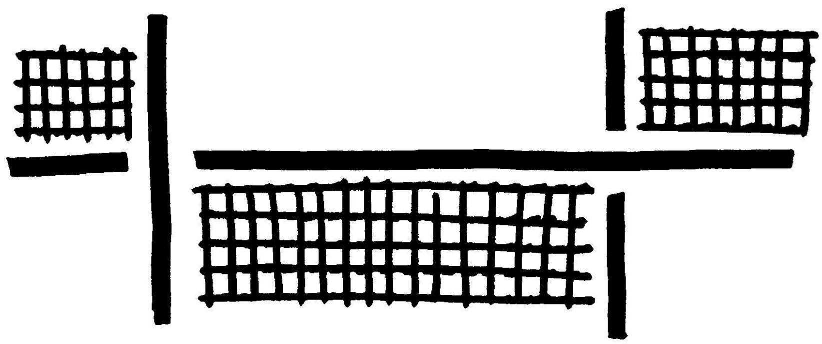

FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram K would not be reduced:

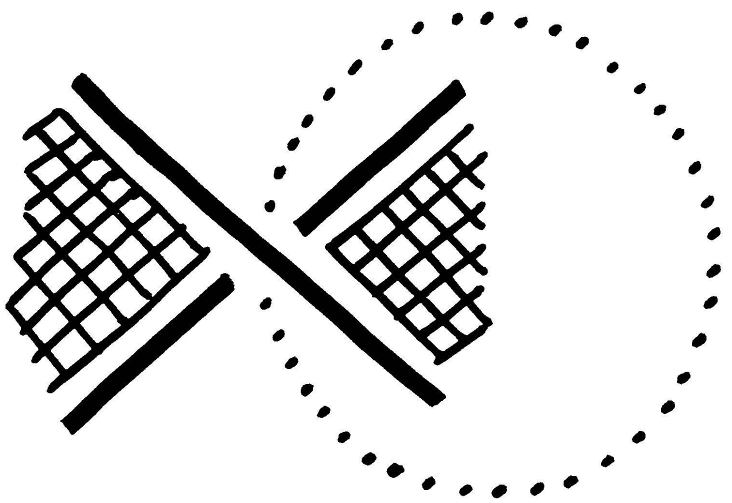

FIGURE 20

It is evident that A is equal to the number of unshaded regions. Let a state S^2 be obtained from A by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore $|S^2| = |A| - 1$. In view of the arguments given in the proof of part (i) of the Theorem, this implies that $D_S < D_A$ for any state S of K. This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

§ 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link L in terms of a (not necessarily orientable) surface V bounded by L (see [2]). One defines a bilinear form