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ON ALTERNATING LINKS
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§ 3. KAUFFMAN'S STATE MODEL FOR THE JONES POLYNOMIAL

Let K be a link diagram. By a state or a marker of K , we mean respectively a state or a marker of the corresponding link projection in R^2 (which is obtained from K by forgetting the overcrossing-undercrossing data). The markers of K are divided into two classes — positive and negative. By definition, if the over-line is rotated counterclockwise around the double point, then the first marker it meets is the positive one and the second one is negative:

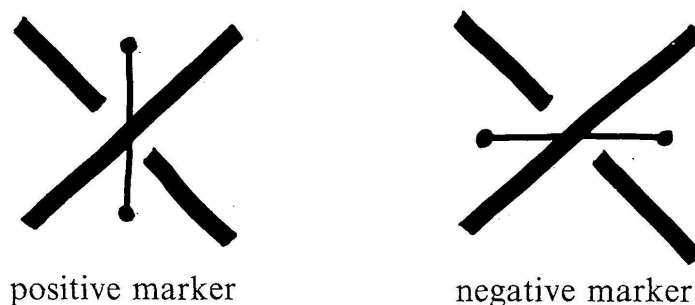


FIGURE 17

Let the diagram K be oriented. Consider the polynomial

$$V_K(t) = (-t)^{-3w(K)/4} \sum t^{(a_S - b_S)/4} (-t^{1/2} - t^{-1/2})^{|S| - 1}$$

where $w(K)$ is the writhe number of K . The summation is over all the states S of K ; the number of positive [respectively negative] markers of the state S is denoted by a_S [respectively b_S], and the number $|S|$ is defined in § 2.

It is shown in [5] that the polynomial $V_K(t)$ is equal to the Jones polynomial of the oriented link presented by K (see also [3]).

§ 4. PROOF OF THEOREM 1

Orient the diagram K and denote the corresponding oriented link by L . Denote by A the state of K in which all markers are positive, and by $B = \check{A}$ the dual state in which all markers are negative. For any state S of K , denote by D_S and d_S respectively the maximal and minimal degrees in t in the expression

$$t^{(a_S - b_S)/4} (-t^{1/2} - t^{-1/2})^{|S| - 1}$$

(see § 3), namely

$$\begin{aligned} D_S &= (a_S - b_S + 2|S| - 2)/4 \\ d_S &= (a_S - b_S - 2|S| + 2)/4. \end{aligned}$$

In particular

$$\begin{aligned} (6) \quad D_A &= (c + 2|A| - 2)/4 \\ d_B &= (-c - 2|B| + 2)/4. \end{aligned}$$

Proof of (i). If a state S^2 is obtained from a state S by replacing one positive marker by a negative one (at some crossing point), then $a_{S^2} = a_S - 1$, $b_{S^2} = b_S + 1$ and $|S^2| \leq |S| + 1$. Thus

$$D_{S^2} - D_S = -\frac{1}{2} + (|S^2| - |S|)/2 \leq 0$$

so that $D_{S^2} \leq D_S$. This implies that $D_S \leq D_A$ for any state S of K . Therefore

$$\begin{aligned} d_{\max}(V_L(t)) &\leq -\frac{3}{4}w(K) + D_A \\ d_{\min}(V_L(t)) &\geq -\frac{3}{4}w(K) + d_B. \end{aligned}$$

Thus in view of equalities (6) and of Lemma 1 of § 2, one has

$$\begin{aligned} (7) \quad \text{span}(L) &\leq D_A - d_B = (c + |A| + |B| - 2)/2 \\ &\leq (2c + 2r - 2)/2 = c + r - 1. \end{aligned} \quad \square$$

Proof of (ii). Let K_1, \dots, K_r be the unsplittable components of K , with $r = r(K)$. Denote by L_i the oriented link represented by K_i . It follows from part (i) of the Theorem and from formula (1) that

$$c(K) = \sum_{i=1}^r c(K_i) \geq \sum_{i=1}^r \text{span}(L_i) = \text{span}(L) - (r - 1).$$

Thus the equality $c(K) + r - 1 = \text{span}(L)$ holds if and only if $c(K_i) = \text{span}(L_i)$ for each i . Therefore, to prove (ii), it suffices to consider the unsplittable case $r = 1$.

It is evident that the numbers $c(K)$ and $\text{span}(L)$ are both additive under connected sum of diagrams. Therefore it is enough to prove the following assertion (*).

- (*) $\left\{ \begin{array}{l} \text{For a prime unsplittable diagram } K \text{ of an oriented link } L, \text{ the} \\ \text{equality } c(K) = \text{span}(L) \text{ holds if and only if } K \text{ is a reduced and} \\ \text{alternating diagram.} \end{array} \right.$

In (*), note that, formally, the link L is not supposed to be prime or even unsplittable.

Suppose first that $c(K) = \text{span}(L)$. Then all inequalities above are in fact equalities. As $r = 1$, one has in particular

$$|A| + |B| = c + 2r = c + 2.$$

Lemma 2 of § 2 shows that the state A is monochrome. This implies that K is alternating, because of the easy but essential lemma:

LEMMA. *Let K be an oriented connected link diagram. Then K is alternating if and only if the state A is monochrome.*

Moreover the diagram K is reduced, since all prime diagrams are reduced except the two diagrams

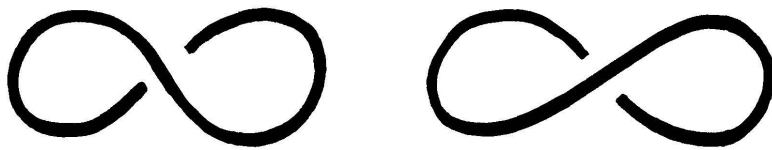


FIGURE 18

which are excluded by the assumption $c(K) = \text{span}(L)$.

Suppose conversely that K is reduced and alternating. The preceding Lemma shows that the state A is monochrome. According to Lemma 2 of § 2: $|A| + |B| = c + 2$. We prove below that

$$(8) \quad d_{\max}(V_L(t)) = -\frac{3}{4}w(K) + D_A$$

$$(9) \quad d_{\min}(V_L(t)) = -\frac{3}{4}w(K) + d_B.$$

Thus the inequalities (7) are in fact equalities, so that $\text{span}(L) = c + r - 1 = c$.

By region, we mean hereafter a connected component of $S^2 - K$. (Here $S^2 = R^2 \cup \{\infty\}$.) Since K is alternating, each region intersects either markers which are all positive or markers which are all negative. Shade the regions of the first type:

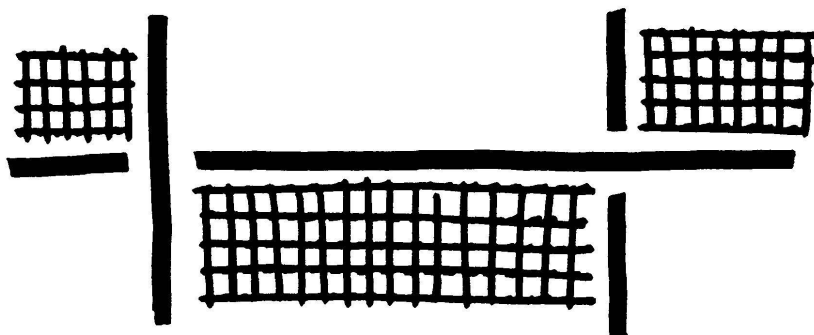


FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram K would not be reduced:

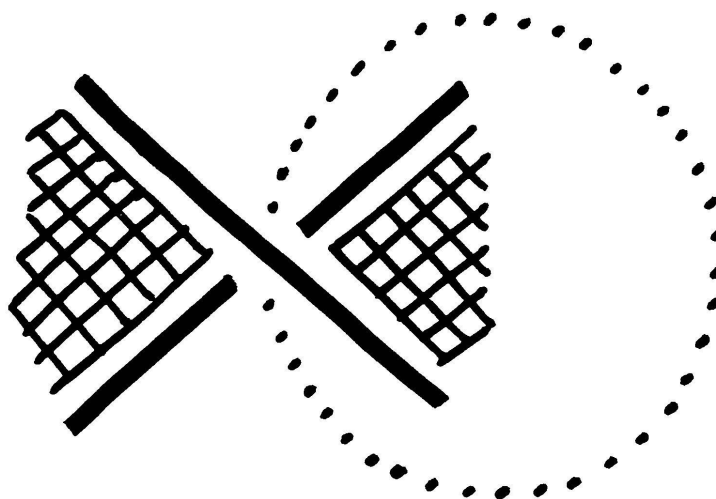


FIGURE 20

It is evident that A is equal to the number of unshaded regions. Let a state S^2 be obtained from A by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore $|S^2| = |A| - 1$. In view of the arguments given in the proof of part (i) of the Theorem, this implies that $D_S < D_A$ for any state S of K . This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

§ 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link L in terms of a (not necessarily orientable) surface V bounded by L (see [2]). One defines a bilinear form