Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 33 (1987)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SIMPLE PROOF OF THE MURASUGI AND KAUFFMAN THEOREMS ON ALTERNATING LINKS

Autor: Turaev, V. G.

Kapitel: §4. Proof of Theorem 1

DOI: https://doi.org/10.5169/seals-87894

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 26.11.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

§ 3. Kauffman's state model for the Jones polynomial

Let K be a link diagram. By a state or a marker of K, we mean respectively a state or a marker of the corresponding link projection in R^2 (which is obtained from K by forgetting the overcrossing-undercrossing data). The markers of K are divided into two classes — positive and negative. By definition, if the over-line is rotated counterclockwise around the double point, then the first marker it meets is the positive one and the second one is negative:

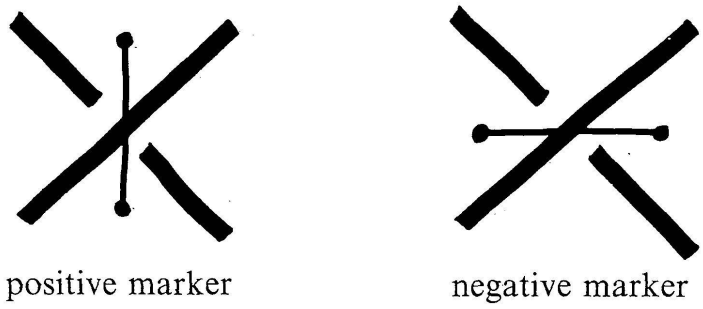

Figure 17

Let the diagram K be oriented. Consider the polynomial

$$V_K(t) = (-t)^{-3w(K)/4} \sum t^{(a_S - b_S)/4}(-t^{1/2} - t^{-1/2})^{|S|-1}$$

where $w(K)$ is the writhe number of K. The summation is over all the states S of K; the number of positive [respectively negative] markers of the state S is denoted by a_S [respectively b_S], and the number $|S|$ is defined in § 2.

It is shown in [5] that the polynomial $V_K(t)$ is equal to the Jones polynomial of the oriented link presented by K (see also [3]).

§ 4. Proof of Theorem 1

Orient the diagram K and denote the corresponding oriented link by L. Denote by A the state of K in which all markers are positive, and by $B = \check{A}$ the dual state in which all markers are negative. For any state S of K, denote by D_S and d_S respectively the maximal and minimal degrees in t in the expression

$$t^{(a_S - b_S)/4}(-t^{1/2} - t^{-1/2})^{|S|-1}$$

(see § 3), namely

$$D_S = (a_S - b_S + 2|S| - 2)/4$$
$$d_S = (a_S - b_S - 2|S| + 2)/4 .$$

In particular

$$D_A = (c + 2|A| - 2)/4$$

(6)

$$d_B = (-c - 2|B| + 2)/4 .$$

Proof of (i). If a state S^2 is obtained from a state S by replacing one positive marker by a negative one (at some crossing point), then $a_{S^2} = a_S - 1$, $b_{S^2} = b_S + 1$ and $|S^2| \leqslant |S| + 1$. Thus

$$D_{S^2} - D_S = -\frac{1}{2} + (|S^2| - |S|)/2 \leqslant 0$$

so that $D_{S^2} \leqslant D_S$. This implies that $D_S \leqslant D_A$ for any state S of K. Therefore

$$d_{\max}(V_L(t)) \leqslant -\frac{3}{4} w(K) + D_A$$

$$d_{\min}(V_L(t)) \geqslant -\frac{3}{4} w(K) + d_B .$$

Thus in view of equalities (6) and of Lemma 1 of § 2, one has

(7)
$$\begin{aligned} \operatorname{span}(L) &\leqslant D_A - d_B = (c + |A| + |B| - 2)/2 \\ &\leqslant (2c + 2r - 2)/2 = c + r - 1 . \end{aligned}$$

□

Proof of (ii). Let $K_1, \ldots, K_r$ be the unsplittable components of K, with $r = r(K)$. Denote by L_i the oriented link represented by K_i. It follows from part (i) of the Theorem and from formula (1) that

$$c(K) = \sum_{i=1}^{r} c(K_i) \geqslant \sum_{i=1}^{r} \operatorname{span}(L_i) = \operatorname{span}(L) - (r-1) .$$

Thus the equality $c(K) + r - 1 = \operatorname{span}(L)$ holds if and only if $c(K_i) = \operatorname{span}(L_i)$ for each i. Therefore, to prove (ii), it suffices to consider the unsplittable case $r = 1$.

It is evident that the numbers $c(K)$ and $\operatorname{span}(L)$ are both additive under connected sum of diagrams. Therefore it is enough to prove the following assertion (*).

(*) For a prime unsplittable diagram K of an oriented link L, the equality $c(K) = \operatorname{span}(L)$ holds if and only if K is a reduced and alternating diagram.

In (*), note that, formally, the link L is not supposed to be prime or even unsplittable.

Suppose first that $c(K) = \operatorname{span}(L)$. Then all inequalities above are in fact equalities. As $r = 1$, one has in particular

$$|A| + |B| = c + 2r = c + 2.$$

Lemma 2 of § 2 shows that the state A is monochrome. This implies that K is alternating, because of the easy but essential lemma:

LEMMA. *Let K be an oriented connected link diagram. Then K is alternating if and only if the state A is monochrome.*

Moreover the diagram K is reduced, since all prime diagrams are reduced except the two diagrams

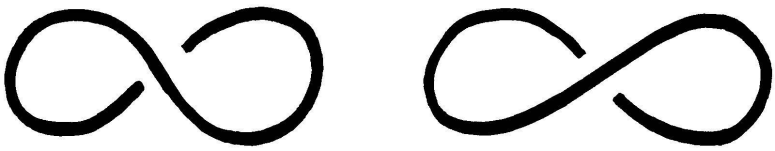

FIGURE 18

which are excluded by the assumption $c(K) = \operatorname{span}(L)$.

Suppose conversely that K is reduced and alternating. The preceeding Lemma shows that the state A is monochrome. According to Lemma 2 of § 2: $|A| + |B| = c + 2$. We prove below that

$$d_{\max}(V_L(t)) = -\frac{3}{4}w(K) + D_A \tag{8}$$

$$d_{\min}(V_L(t)) = -\frac{3}{4}w(K) + d_B. \tag{9}$$

Thus the inequalities (7) are in fact equalities, so that $\operatorname{span}(L) = c + r - 1 = c$.

By region, we mean hereafter a connected component of $S^2 - K$. (Here $S^2 = R^2 \cup \{\infty\}$.) Since K is alternating, each region intersects either markers which are all positive or markers which are all negative. Shade the regions of the first type:

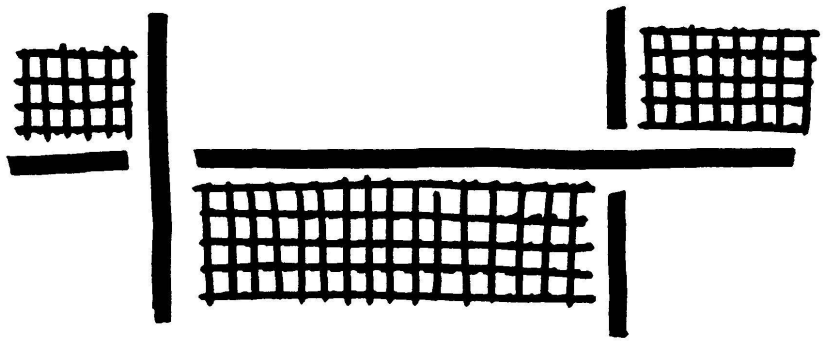

FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram K would not be reduced:

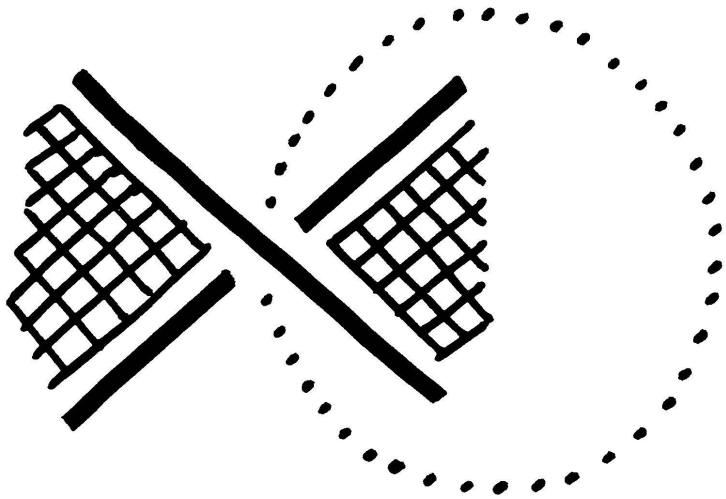

FIGURE 20

It is evident that A is equal to the number of unshaded regions. Let a state S^2 be obtained from A by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore $|S^2| = |A| - 1$. In view of the arguments given in the proof of part (i) of the Theorem, this implies that $D_S < D_A$ for any state S of K. This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

§ 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link L in terms of a (not necessarily orientable) surface V bounded by L (see [2]). One defines a bilinear form