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THE VARIETY OF MODULI OF FOLIATIONS ON A COMPLEX SPACE

by Hans-Jörg REIFFEN

§0. Let X be a complex space. By X_s we denote the singular locus of X . Let \mathcal{F} be a coherent analytic sheaf on X . By $S(\mathcal{F})$ we denote the singular locus of \mathcal{F} , i.e. the set of points, where \mathcal{F} is not free. $S(\mathcal{F})$ is an analytic subset of X . If X is reduced then $S(\mathcal{F})$ is thin. For a coherent subsheaf \mathcal{F}' of \mathcal{F} we set $S(\mathcal{F} : \mathcal{F}') := S(\mathcal{F}) \cup S(\mathcal{F}/\mathcal{F}')$.

In the following let X be a reduced complex space. If X' is an irreducible component of X , then we denote by $r(\mathcal{F}, X')$ the rank of \mathcal{F} on $X' \setminus (X_s \cup S(\mathcal{F}))$. Ω resp. Θ is the sheaf of holomorphic 1-forms resp. of vector fields on X . We have $S(\Omega) = X_s$.

0.1 DEFINITION. Let Ω' be a coherent subsheaf of Ω and X' an irreducible component of X . Ω' is a X' -foliation, iff there is a thin analytic subset A of X' , $X' \cap X_s \subset A$, such that $\Omega' | X' \setminus A$ defines a regular foliation. Ω' is a foliation, iff Ω' is a X' -foliation for every irreducible component X' of X .

In a joint work with G. Bohnhorst (comp. [B/R]) I have developed a general theory of foliations. The coherent foliations as they are introduced in [B/R] correspond to those foliations used in this paper, which are full subsheaves of Ω . We call $\text{codim}_{X'} \Omega' := r(\Omega'; X')$ the codimension of the X' -foliation.

The following homomorphism of sheaves (of groups) is important

$$\delta^p : \Omega^{p+1} \rightarrow \bigwedge^{p+2} \Omega, \quad \delta^p(\omega^0; \omega^1, \dots, \omega^p) := d\omega^0 \wedge \omega^1 \wedge \dots \wedge \omega^p.$$

0.2 REMARK. Let Ω' be a coherent subsheaf of Ω . Then the following are equivalent:

- (1) Ω' is a X' -foliation of codimension p ;
- (2) $p = r(\Omega'; X')$, $\delta^p | (\Omega' | X' \setminus X_s)^{p+1} = 0$;
- (3) $p = r(\Omega'; X')$, there is a point $x \in X' \setminus X_s$, such that $\delta^p((\Omega'_x)^{p+1}) = 0$.

For the following compilation of definitions and results the paper by Douady [Dou] is a good reference.

In the following let Y, Z be further complex spaces, that are not necessarily reduced. If \mathcal{E} is a coherent analytic sheaf on Y and $f: Z \rightarrow Y$ a holomorphic mapping then we denote by $f^*\mathcal{E}$ the analytic inverse image of \mathcal{E} on Z .

Let \mathcal{E} be a coherent analytic sheaf on $Y \times X$. For $y \in Y$ we denote by $\mathcal{E}(y)$ the analytic restriction of \mathcal{E} on $X = y \times X$; $\mathcal{E}(y)$ is the analytic inverse image of \mathcal{E} to the injection $X = y \times X \hookrightarrow Y \times X$. If $f: Z \rightarrow Y$ is a holomorphic mapping then we set $\mathcal{E}_Z := (f \times \text{id}_X)^*\mathcal{E}$. We have $\mathcal{E}_Z(z) = \mathcal{E}(f(z))$ for $z \in Z$.

Let the coherent analytic sheaf \mathcal{E} on $Y \times X$ be Y -flat, then $r(\mathcal{E}(y), X')$ is locally constant on Y for every irreducible component X' of X . Therefore

$$R_{X'}^p(Y, \mathcal{E}) := \{y \in Y : r(\mathcal{E}(y), X') = p\}$$

is an open and closed subset of Y for every p . Let $\check{\mathcal{R}}$ be a coherent subsheaf of \mathcal{E} , so that $\mathcal{R} := \mathcal{E}/\check{\mathcal{R}}$ is also Y -flat. Then $\check{\mathcal{R}}$ is Y -flat too and we have $\mathcal{R}(y) = \mathcal{E}(y)/\check{\mathcal{R}}(y)$ in a natural way. Let $f: Z \rightarrow Y$ be a holomorphic mapping. Then \mathcal{E}_Z and \mathcal{R}_Z are Z -flat and we have $\mathcal{R}_Z = \mathcal{E}_Z/\check{\mathcal{R}}_Z$ in a natural way.

Let \mathcal{F} be a coherent analytic sheaf on X . Then the analytic inverse image $\mathcal{F}^Y := \pi^*\mathcal{F}$ to the projection $\pi: Y \times X \rightarrow X$ is Y -flat and we have $\mathcal{F}^Y(y) = \mathcal{F}$ in a natural way. In the following let $\check{\mathcal{R}}$ be a coherent subsheaf of Ω^Y , such that $\mathcal{R} := \Omega^Y/\check{\mathcal{R}}$ is Y -flat. Let X' be an irreducible component of X . We set:

$$F_{X'}^p(Y, \check{\mathcal{R}}) := \{y \in Y : \check{\mathcal{R}}(y) \text{ is a } X'\text{-foliation of codimension } p\},$$

$$F_{X'}(Y, \check{\mathcal{R}}) := \{y \in Y : \check{\mathcal{R}}(y) \text{ is a } X'\text{-foliation}\},$$

$$F(Y, \check{\mathcal{R}}) := \{y \in Y : \check{\mathcal{R}}(y) \text{ is a foliation}\}.$$

We will show:

0.3 THEOREM. $F_{X'}^p(Y, \check{\mathcal{R}})$ is an analytic subset of Y .

By 0.3 we get:

0.4 COROLLARY.

- (1) $F_{X'}(Y, \check{\mathcal{R}})$ is an analytic subset of Y ;
- (2) $F(Y, \check{\mathcal{R}})$ is an analytic subset of Y .

Because \mathcal{R} is Y -flat we get the following:

0.5 REMARK. $S(\Omega^Y: \check{\mathcal{R}}) \cap (y \times X) = S(\Omega: \check{\mathcal{R}}(y))$.

In 0.5 we identify $X = y \times X$.

Let $f: Z \rightarrow Y$ be a holomorphic mapping. Then $\Omega_Z^Y = \Omega^Z$ and we obviously get:

0.6 REMARK. $F_{X'}^p(Z, \check{\mathcal{R}}_Z) = f^{-1}(F_{X'}^p(Y, \check{\mathcal{R}}))$.

$F_{X'}^p(Y, \check{\mathcal{R}})$ is an analytic subset of $R_{X'}^p(Y, \check{\mathcal{R}})$. Using the homomorphism δ^p we construct a canonical complex structure on $F_{X'}^p(Y, \check{\mathcal{R}})$ and show:

0.7 THEOREM. *The holomorphic mapping $f: Z \rightarrow Y$ induces a holomorphic mapping $F_{X'}^p(Z, \check{\mathcal{R}}_Z) \rightarrow F_{X'}^p(Y, \check{\mathcal{R}})$ related to the canonical complex structures.*

We can apply $F(Y, \check{\mathcal{R}})$ with the structure given by the spaces $F_{X'}^p(Y, \check{\mathcal{R}})$. We get the following (comp. [Dou]):

0.8 COROLLARY. *Let X be compact and let H be the Douady space of Ω, \mathcal{R} the universal Douady sheaf of Ω . Then the complex subspace $F(H, \check{\mathcal{R}})$ of H has the following property:*

If Z is a complex space and $\mathcal{S} = \Omega^Z/\check{\mathcal{S}}$ a coherent and Z -flat sheaf, then the unique holomorphic mapping $f: Z \rightarrow H$ with $\mathcal{S} = \check{\mathcal{R}}_Z$ factorizes over $F(Z, \check{\mathcal{S}}), F(H, \check{\mathcal{R}})$.

In this sense $F(H, \check{\mathcal{R}})$ is a moduli space of the foliations on X .

0.9 PROPOSITION. *Let X be a connected compact manifold. Then the following sets are complements of analytic subsets in $F(Y, \check{\mathcal{R}})$:*

$$F_r(Y, \check{\mathcal{R}}) := \{y \in Y: \check{\mathcal{R}}(y) \text{ is a regular foliation}\},$$

$$F_f(Y, \check{\mathcal{R}}) := \{y \in F(Y, \check{\mathcal{R}}): \check{\mathcal{R}}(y) \text{ is locally free}\},$$

$${}^kF(Y, \check{\mathcal{R}}) := \{y \in F(Y, \check{\mathcal{R}}): \text{codim } S(\Omega: \check{\mathcal{R}}(y)) \geq k\}.$$

$F_r(Y, \check{\mathcal{R}}) \cap {}^2F(Y, \check{\mathcal{R}})$ is the set of points $y \in Y$ such that $\check{\mathcal{R}}(y)$ is a free foliation in the sense of [B/R]. By the theorem of Frobenius-Malgrange all foliations $\check{\mathcal{R}}(y), y \in F_r(Y, \check{\mathcal{R}}) \cap {}^3F(Y, \check{\mathcal{R}})$ are locally integrable.

In an earlier version of this paper ([Re]) I gave a proof of 0.3 by Banach analytic technics. With these technics G. Pourcin has gotten results similar to those in this paper ([Pou]). Similar results were also proved by X. Gomez-Mont ([GM]). He considers foliations defined by vector fields; his technics are closer to those of this paper. The advantage of my approach is the simplicity. For 0.3, the canonical complex structure on $F_{X'}^p(Y, \check{\mathcal{R}}), F(Y, \check{\mathcal{R}})$ and 0.7 we need no compactness.

§ 1. We prove 0.3: Because 0.3 is a local theorem related to Y and because of 0.2 we may assume the following:

Y is an analytic subspace of a polycylinder $P = \{s \in \mathbf{C} : |s| < \sigma\}^m$ in \mathbf{C}^m defined by an ideal sheaf \mathcal{I} . We denote the coordinates of \mathbf{C}^m by $y = (y_1, \dots, y_m)$. $Y = R_{\mathcal{I}}^p(Y, \mathcal{R})$.

X is a polycylinder $\{t \in \mathbf{C} : |t| < \tau\}^n$ in \mathbf{C}^n . We denote the coordinates of \mathbf{C}^n by $x = (x_1, \dots, x_n)$. $S(\Omega^Y : \mathcal{R}) = \emptyset$.

We can interpret $(\overset{r}{\wedge} \Omega)^P$ as a submodule of $\overset{r}{\wedge} \Omega_{P \times X}$: $(\overset{r}{\wedge} \Omega)^P$ consists of all r -forms η of the form $\eta = \sum_{1 \leq v_1 < \dots < v_r \leq n} \eta_{v_1 \dots v_r}(y, x) dx_{v_1} \wedge \dots \wedge dx_{v_r}$.

We notice this by $\eta = \eta(y, x; dx)$. We get $(\overset{r}{\wedge} \Omega)^Y = (\overset{r}{\wedge} \Omega)^P / \mathcal{I} \cdot (\overset{r}{\wedge} \Omega)^P$.

We may assume, that there are forms $\eta^1, \dots, \eta^q \in \Gamma(P \times X, \Omega^P)$ generating \mathcal{R} everywhere. We shorten $F := F_{\mathcal{I}}^p(Y, \mathcal{R})$. For $y^0 \in Y$ we get:

$$y^0 \in F \Leftrightarrow \delta^p(\eta^{j_0}(y^0, x; dx); \eta^{j_1}(y^0, x; dx), \dots, \eta^{j_p}(y^0, x; dx)) = 0$$

for all $1 \leq j_0, j_1, \dots, j_p \leq q$.

We consider an arbitrary system $\omega^0, \omega^1, \dots, \omega^p \in \Gamma(P \times X, \Omega^P)$. By fixing y we define

$$\mathfrak{P}^p(\omega^0; \omega^1, \dots, \omega^p)$$

$$:= \delta^p(\omega^0(y, x; dx); \omega^1(y, x; dx), \dots, \omega^p(y, x; dx)) \in \Gamma(P \times X, (\overset{p+2}{\wedge} \Omega)^P).$$

We have

$$\mathfrak{P}^p(\omega^0; \omega^1, \dots, \omega^p) = \sum_{1 \leq v_1 < \dots < v_{p+2} \leq n} A_{v_1 \dots v_{p+2}} dx_{v_1} \wedge \dots \wedge dx_{v_{p+2}}$$

with holomorphic coefficients

$$A_{v_1 \dots v_{p+2}} = A_{v_1 \dots v_{p+2}}(\omega^0; \dots, \omega^p) \in \Gamma(P \times X, \mathcal{O}_{P \times X}).$$

We form the serial expansions

$$A_{v_1 \dots v_{p+2}} = \sum_{l \in \mathbf{N}^n} A_{v_1 \dots v_{p+2}; l} x^l$$

with holomorphic coefficients $A_{v_1 \dots v_{p+2}; l} = A_{v_1 \dots v_{p+2}; l}(\omega^0, \dots, \omega^p) \in \Gamma(P, \mathcal{O}_P)$.

With these notations we get:

The set F is defined on Y by the following (infinite) system of holomorphic functions:

$$A_{v_1 \dots v_{p+2}; l}(\eta^{j_0}; \eta^{j_1}, \dots, \eta^{j_p});$$

$$1 \leq j_0, j_1, \dots, j_p \leq q, 1 \leq v_1 < \dots < v_{p+2} \leq n, l \in \mathbf{N}^n.$$

F is an analytic subset of Y .

§ 2. We refine the considerations of § 1. We can interpret \mathfrak{P}^p as a homomorphism of sheaves (of groups) $\mathfrak{P}^p: (\Omega^P)^{p+1} \rightarrow (\wedge^{p+2} \Omega)^P$. Obviously \mathfrak{P}^p induces a homomorphism $\mathfrak{P}_Y^p: (\Omega^Y)^{p+1} \rightarrow (\wedge^{p+2} Y)$, which does not depend on the representation of Y as a subspace of a number space. Let \mathcal{I} be the ideal sheaf on P , defined by \mathcal{I} and the functions $A_{v_1 \dots v_{p+2}; l}(\eta^{j_0}; \eta^{j_1}, \dots, \eta^{j_p})$ (comp. [Fr]). We apply F with the structure sheaf $\mathcal{H} := \mathcal{O}_P / \mathcal{I}$ and consider the natural injection $\iota: F \rightarrow Y$.

2.1 PROPOSITION. \mathcal{H} is the maximal complex structure on F , such that $\mathfrak{P}_{(F, \mathcal{H})}^p | (\tilde{\mathcal{R}}_{(F, \mathcal{H})})^{p+1} = 0$, i.e.

- (1) $\mathfrak{P}_{(F, \mathcal{H})}^p | (\tilde{\mathcal{R}}_{(F, \mathcal{H})})^{p+1} = 0$;
- (2) if $(F, \tilde{\mathcal{H}})$ is an analytic subspace of Y , such that $\mathfrak{P}_{(F, \tilde{\mathcal{H}})}^p | (\tilde{\mathcal{R}}_{(F, \tilde{\mathcal{H}})})^{p+1} = 0$, then $(F, \tilde{\mathcal{H}})$ is an analytic subspace of (F, \mathcal{H}) .

Proof. (1) Let $\omega^0, \omega^1, \dots, \omega^p \in \Gamma(P \times X, \Omega)$ induce elements of $\Gamma(F \times X, \tilde{\mathcal{R}}_H)$. We may assume, that there are representations

$$\omega^k = \sum_{j=1}^q a_{kj} \eta^j \text{ mod } \Gamma(P \times X, \mathcal{I} \cdot \Omega^p).$$

Because of

$$\eta^{j_0} \wedge \eta^{j_1} \wedge \dots \wedge \eta^{j_p} \in \Gamma(P \times X, \mathcal{I} \cdot (\wedge^{p+1} \Omega)^P)$$

we get mod $\Gamma(P \times X, \mathcal{I} \cdot (\wedge^{p+2} \Omega)^P)$:

$$\mathfrak{P}^p(\omega^0; \omega^1, \dots, \omega^p) = \sum_{1 \leq j_0, \dots, j_p \leq q} a_{0j_0} \dots a_{pj_p} \mathfrak{P}^p(\eta^{j_0}; \eta^{j_1}, \dots, \eta^{j_p}).$$

Therefore we get

$$A_{v_1 \dots v_{p+2}; l}(\omega^0; \omega^1, \dots, \omega^p) \in \Gamma(P, \mathcal{I}), \mathfrak{P}^p(\omega^0; \omega, \dots, \omega^p) \in \Gamma(P \times X, \mathcal{I} \cdot (\wedge^{p+2} \Omega)^P).$$

(2) Let $\tilde{\mathcal{I}}$ be the ideal sheaf of \mathcal{O}_P defining $\tilde{\mathcal{H}}$. We show $\mathcal{I} \subset \tilde{\mathcal{I}}$. We have

$$\mathfrak{P}^p(\eta^{j_0}; \eta^{j_1}, \dots, \eta^{j_p}) \in \Gamma(P \times X, \tilde{\mathcal{I}} \cdot (\wedge^{p+2} \Omega)^P)$$

and therefore

$$A_{\nu_1 \dots \nu_{p+2}, l}(\eta^{j_0}; \eta^{j_1}, \dots, \eta^{j_p}) \in \Gamma(P, \tilde{\mathcal{I}}).$$

Now we consider an arbitrary reduced complex space X and an arbitrary complex space Y . We again set $F := F_X^p(Y, \tilde{\mathcal{R}})$.

Let $y^0 \in F$. We consider $X'(y^0) := X \setminus S(\Omega: \tilde{\mathcal{R}}(y^0))$. If $x^0 \in X'(y^0)$ then we can realize the situation of § 1 related to open neighbourhoods V_0 of y^0 and U_0 of x^0 . Let $\mathcal{H}(x^0)$ denote the structure sheaf of $V_0 \cap F$ according to 2.1. We get $\mathcal{H}(x^0)_{y^0} = \mathcal{H}(x)_{y^0}$ for every $x \in U_0$. Because $X'(y^0)$ is connected, we get $\mathcal{H}(x^0)_{y^0} = \mathcal{H}(x)_{y^0}$ for every $x \in X'(y^0)$. Let \mathcal{H} be the structure sheaf on F defined by $\mathcal{H}_y := \mathcal{H}(x)_y, x \in X'(y)$.

If $\tilde{\mathcal{H}}$ is any structure sheaf on F such that $(F, \tilde{\mathcal{H}})$ is an analytic subspace of Y we get by 0.5

$$S(\Omega^{(F, \tilde{\mathcal{H}})}: \tilde{\mathcal{R}}_{(F, \tilde{\mathcal{H}})}) = S(\Omega^Y: \tilde{\mathcal{R}}) \cap (F \times X).$$

We set

$$(F \times X)^0 := (F \times X) \setminus S(\Omega^Y: \tilde{\mathcal{R}}).$$

By 2.1 we get:

2.2 PROPOSITION. \mathcal{H} is the maximal complex structure on F , such that $\mathfrak{D}_{(F, \mathcal{H})}^p | (\tilde{\mathcal{R}}_{(F, \mathcal{H})} | (F \times X)^0)^{p+1} = 0$.

We prove 0.7: We may assume that Z, X and Y, X fulfill the assumptions of § 1 simultaneously. We denote the objects related to Z by the index Z . Further we may assume, that f is given by a holomorphic mapping $g: P_Z \rightarrow P$ with $g^*\mathcal{I} \subset \mathcal{I}_Z$. We consider $\omega^j(z, x; dx) := \eta^j(g(z), x; dx)$. Then ω^j generates an element of $\Gamma(Z \times X, \tilde{\mathcal{R}}_Z)$ and we get

$$\begin{aligned} & A_{\nu_1 \dots \nu_{p+2}, l}(\eta^{j_0}; \eta^{j_1}, \dots, \eta^{j_p}) \circ (f \times \text{id}_X) \\ &= A_{\nu_1 \dots \nu_{p+2}, l}(\omega^{j_0}; \omega^{j_1}, \dots, \omega^{j_p}) \in \Gamma(P_Z, \mathcal{I}_Z). \end{aligned}$$

We prove 0.9: Let $\pi: Y \times X \rightarrow Y$ be the projection then $\pi(S(\Omega^Y: \tilde{\mathcal{R}}))$ and $\pi(S(\tilde{\mathcal{R}}))$ are analytic subsets of Y . $\{(y, x) \in Y \times X: \dim_{(y, x)}(S(\Omega^Y: \tilde{\mathcal{R}}) \cap (y \times X)) > \dim X - k\}$ is analytic and therefore $\{y \in Y: \text{codim } S(\Omega: \tilde{\mathcal{R}}(y)) < k\}$ too.

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