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Artikel: ON THE GALOIS GROUPS OF THE EXPONENTIAL TAYLOR POLYNOMIALS

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Kapitel: VI. Final Remarks

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IV. Calculation of the Discriminant

Let $f_n(x) = \frac{1}{n!}(x-\alpha_1)\dots(x-\alpha_n)$. Then

$$\begin{aligned} D_n &= \prod_{i\neq j}(\alpha_i-\alpha_j) = (-1)^{\binom{n}{2}}\prod_{i<j}(\alpha_i-\alpha_j) \\ &= (-1)^{\binom{n}{2}}\prod_{i=1}^{n}(n!\,f_n'(\alpha_i)) = (-1)^{\binom{n}{2}}\prod_{i=1}^{n} n!\,f_{n-1}(\alpha_i) \\ &= (-1)^{\binom{n}{2}}\prod_{i=1}^{n}(-\alpha_i^n) \qquad (\text{since } f_{n-1}(x) = f_n(x) - \frac{x^n}{n!}) \\ &= (-1)^{\binom{n}{2}+n}(\prod_{i=1}^{n}\alpha_i)^n = (-1)^{\binom{n}{2}+n}((-1)^n n!\,f_n(0))^n \\ &= (-1)^{\binom{n}{2}}(n!)^n . \end{aligned}$$

V. End of Proof

Tschebyshev's Theorem (2 above) implies that for each $n > 1$ there is a prime number p such that

$$\mathrm{ord}_p(n!) = 1 .$$

Hence D_n is not a square if n is odd. If $n \equiv 2(4)$ then $D_n < 0$ so it is not a square in this case either. Finally if 4 divides n then we see that D_n is a square. This completes the proof for $n \geqslant 8$. The remaining cases can be handled individually using the above results and facts about S_n for small n. (See [S-2].)

VI. Final Remarks

1. Hilbert [H] proved that there exist extensions of $\mathbf{Q}$ with Galois group S_n. The splitting fields of the exponential polynomials provide explicit examples of such extensions. Moreover, they provide examples of such extensions ramified only at the primes dividing the order of the Galois group, a property not predictable by Hilbert's methods. (In fact, as can easily be checked using the results of II above, they are ramified at all primes dividing the order of the Galois group.) Schur also found A_n extensions of $\mathbf{Q}$ for n odd unramified outside $n!$ (see [S-2] and [S-3]). This raises the question, given a simple group G, does there exist a G extension of $\mathbf{Q}$ unramified outside the order of G?

2. The original proof of Schur utilized the following result:

THEOREM (Schur 1929 [S-1]). *Let* $1 \leqslant k \leqslant h$ *be integers. Then there exists a prime number* $p > k$ *which divides one of the following integers:*

$$h+1,\ h+2, \ldots, h+k .$$

The proof uses Tschebyshev's method. Schur needed this result to demonstrate the irreducibility of f_n, which we were able to obtain by elementary means. However, Schur obtained much more. He proved:

THEOREM (Schur, 1929, [S-1]). *Let* $a_0, a_1, \ldots, a_n$ *be integers such that* $(a_0, a_n, n!) = 1$. *Then*

$$a_0 + a_1 x + \frac{a_2 x^2}{2!} + \ldots + \frac{a_n x^n}{n!}$$

is irreducible.

3. *Exercises.*

(a) Calculate the Galois Groups of the Following polynomials:

(1) (Laguerre) $$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!},$$

(2) $$J_n(x) = \frac{1}{x} \int_0^x L_n(t)dt = \frac{1}{n+1} \frac{dL_{n+1}(x)}{dx},$$

(3) (Hermite) $$H_m(x) = \sum_{k=0}^{[m/2]} \left(-\frac{1}{2}\right)^k \frac{m!}{(m-2k)!\,k!} x^{m-2k},$$

$$K_n^{(0)}(x^2) = H_{2n}(x), \quad K_n^{(1)}(x^2) = \frac{1}{x} H_{2n+1}(x).$$

(b) Calculate the discriminants of polynomials:

$$\sum_{v=0}^{n} \binom{n+\alpha}{n-v} \frac{(-x)^v}{v!}, \qquad \text{for} \quad \alpha \in \mathbf{Q}.$$

(See [S-2] and [S-3].) Using either Schur's Criterion above or the methods of this note, determine the irreducibility of as many of these polynomials as you can.

(c) For each prime p, determine the inertia subgroups of G_n above p. (Note, we have only done this when n is a power of p.)