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 $c_1(L) \ge 0$  on M and  $c_1(L) > 0$  outside some subvariety of M. Siu [S5] proved that the converse is also true under the weaker assumption that  $c_1(L)$  is nonnegative everywhere and positive at some point. Thus, a manifold which satisfies (\*\*) is Moishezon. It is also not known whether  $\mathbb{CP}^n$ ,  $n \ge 4$ , can admit a nonstandard structure which is Moishezon. For n = 3, T. Peternell [Pe] proved that if M is a Moishezon 3-fold which is topologically isomorphic to  $\mathbb{CP}^3$ , then M is the standard  $\mathbb{CP}^3$ . His proof depends heavily on Mori's theory of extremal rays in 3-folds. One might expect that it is helpful for this problem to study rational curves in a Moishezon manifold which is a topological  $\mathbb{CP}^n$ .

# 5. Kähler-Einstein Metrics on Noncompact Manifolds

We now consider Kähler-Einstein metrics on complete noncompact manifolds. Let g be a complete Kähler-Einstein metric on  $M^n$ , i.e.,  $R_{ij} = cg_{ij}$ for some constant c. If c > 0, Myer's theorem would imply N is compact. Hence,  $c \leq 0$  and  $c_1(M) \leq 0$ . In this section we consider the case  $c_1(M) < 0$ and leave the case  $c_1(M) = 0$  for the next section.

One would like to characterize noncompact manifolds which admit complete Kähler-Einstein metrics  $g_{ij}$  with  $R_{ij} = -g_{ij}$ . In particular, one would like to impose conditions on M to guarantee the existence and uniqueness of a Kähler-Einstein metric. First of all, uniqueness always holds. That is to say, if M and N are complete Kähler-Einstein manifolds with R = -1and  $F: M \to N$  is a biholomorphism, then F is an isometry. To prove this, let g and dv and g' and dv' denote the Kähler-Einstein metrics and volume forms of M and N, respectively. If we let  $\rho = \log (F^* dv'/dv)$ , then  $\partial \overline{\partial} \rho$  $= -f^* \operatorname{Ric}' + \operatorname{Ric} = F^* g' + g$ . Taking traces, we have  $\Delta \rho = -n$  $+ n \cdot e^{\rho/n}$ . Hence, the maximum principle implies  $\rho \leq 0$  and  $F^* dv' \leq dv$ . Replacing F by  $F^{-1}$ , we have  $F^* dv' \geq dv$  and F is an isometry.

Uniqueness also holds for "almost" complete Kähler-Einstein metrics with scalar curvature equal to minus one. Here, a metric  $ds^2$  on M is said to be almost complete if we can write M as an increasing union of domains  $\Omega_{\alpha}$ and there exist complete metrics  $ds_{\alpha}^2$  on  $\Omega_{\alpha}$  for each  $\alpha$  such that  $ds_{\alpha}^2$  converges to  $ds^2$  on compact subsets of M. See Cheng-Yau [C-Y1] for details.

We now consider the existence of Kähler-Einstein metrics with negative scalar curvature. Of course, the existence of such a metric would give restrictions on the complex structure of M. For example, Eiseman [Ei] proved that if there exists a Hermitian metric with scalar curvature less than

a negative constant on M, then the pseudomeasure in the sense of Eiseman is in fact a measure, that is to say, M is measure hyperbolic.

In [C-Y1], Cheng and Yau obtained the existence of Kähler-Einstein metrics on a large class of noncompact manifolds. More precisely, they proved the following. Let  $M^n$  be a Hermitian manifold whose Ricci tensor defines a Kähler metric whose curvature and its covariant derivatives are bounded. Then M admits a Kähler-Einstein metric which is uniformly equivalent to the above metric.

If M admits a Hermitian metric with strongly negative Ricci curvature and is the increasing union of relatively compact, smooth, pseudoconvex open submanifolds, then there exists a unique (up to a scalar) almost complete Kähler-Einstein metric on M. Moreover, this metric is complete if M is complete.

In particular, there exists a complete Kähler-Einstein metric on any bounded domain in  $\mathbb{C}^n$  which is the intersection of domains with  $C^2$ boundaries. In the above statement,  $\mathbb{C}^n$  can also be replaced by a Hermitian manifold with Ricci curvature bounded from above by a negative constant.

Mok and Yau [Mk-Y] proved that there exists a complete Kähler-Einstein metric on any bounded pseudoconvex domain in  $\mathbb{C}^n$ . This is the only known "canonical" metric on arbitrary bounded domains of holomorphy which is complete.

We now consider the case where the volume of M is finite. In this case, the "infinity" of M is very small (whereas the infinity of a bounded domain in  $\mathbb{C}^n$  is quite large). The following is then conjectured: If the Ricci curvature is negative and M has finite topological type, then M can be compactified, that is,  $M = \overline{M}/(\text{subvariety})$  for some compact Kähler manifold  $\overline{M}$ . In some cases,  $\overline{M}$  is actually algebraic and hence M is quasi-projective.

For a locally Hermitian symmetric space M of finite volume, Baily and Borel [B-B], Satake [St] and Mumford [Mu] obtained (different) compactifications more or less explicitly. For these manifolds, Kähler-Einstein metrics exist. Siu and Yau [S-Y3] proved that a complete manifold, with finite volume with its curvature bounded between two negative constants, is quasiprojective.

If the above conjecture is true, then in studying Kähler manifolds with finite volume (and bounded covariant derivatives of the curvature) one need only consider  $\overline{M} \setminus (D_1 \cup \cdots \cup D_k)$  where  $\overline{M}$  is a compact Kähler manifold and  $D_1, \ldots, D_k$  are connected divisors. If we have suitable algebraic data on how  $D_i$  looks like and how  $D_i$  intersects  $D_j$ , then one hopes that one may be able to construct Kähler-Einstein metrics on M. In dimension two, this is well understood. For example, suppose  $C \subseteq \overline{M}^2$  is an elliptic curve and  $C \cdot C < 0$ . If s is a section of the bundle [C] and  $C = \{s=0\}$  then  $dv_{\overline{M}}/|s|^2(\log|s|^2)^3$  is a complete asymptotic Kähler-Einstein metric on  $\overline{M}/C$  with C as the cusps of the metric.

Suppose that D is a divisor on a compact Kähler manifold M satisfying  $c_1(K+[D]) \ge 0$  on  $\overline{M}$ ,  $c_1(K+[D]) > 0$  on  $\overline{M} \setminus D$  and  $(K+[D]) - \varepsilon[D]|_D > 0$  then  $\overline{M} \setminus D$  admits a Kähler-Einstein metric with finite volume. Moreover, the curvature of the metric and all of its covariant derivatives are bounded. It is not clear whether complete Kähler-Einstein metrics should have bounded curvature.

For a quasi-projective manifold  $M = \overline{M} \setminus D$ , a Kähler-Einstein metric always has finite volume and one can define logarithmic Chern classes  $\tilde{c}_i(M, D)$ . The existence of the Kähler-Einstein metric implies the following inequality for the log Chern classes  $\tilde{c}_1$  and  $\tilde{c}_2$ :

$$(*') \qquad (-1)^n \tilde{c}_1^{n-2} \cdot \tilde{c}_2 \ge \frac{(-1)^n}{2(n+1)} \, \tilde{c}_1^n \, .$$

A particularly significant fact is that equality holds in (\*) if the quasiprojective manifold  $\overline{M} \setminus D$  is the quotient of the unit ball in  $\mathbb{C}^n$ .

Recall that a complex manifold is called measure hyperbolic if the Kobayashi measure is positive everywhere. Moreover, for a complete Kähler-Einstein manifold, the following inequality holds,

$$c_1 dv_{\text{Kobayashi}} \ge dv_{\text{Kähler-Einstein}} \ge c_2 dv_{\text{Caratheodory}}$$

where  $c_1$  and  $c_2$  are two universal positive constants. We have the following question: If the Caratheodory metric of M is complete, does M admit a complete Kähler-Einstein metric?

## 6. RICCI FLAT METRICS ON NONCOMPACT MANIFOLDS

We now consider Ricci flat metrics on a complete, noncompact manifold M. We first remark that in this case uniqueness is unknown. Even for compact manifolds, Kähler-Einstein metrics are only unique in each Kähler class. Suppose g and g' are two Ricci flat Kähler metrics on M. If they satisfy  $g_{ij} - g'_{ij} = \partial \overline{\partial} F$  with F bounded, then  $g_{ij} = g'_{ij}$ . Note that in the compact case, the above condition means that g and g' belong to the same Kähler class. It also may be possible to drop the condition that F is bounded since there do not exist too many Ricci flat metrics.