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3. Uniformization

In the one complex dimensional case, we know that every Riemannian surface is one of the following:

$\mathbf{CP}^1$: the Riemannian sphere, which has a unique complex structure,

E: an elliptic curve, which is covered holomorphically by $\mathbf{C}$,

$\Sigma_g, (g>1)$: a surface covered holomorphically by the unit disk $D \subseteq \mathbf{C}$.

In higher dimensions, many results and classifications come from trying to generalize the above classification. One wants to know under what geometric conditions is M biholomorphic to a higher dimensional analogue of $\mathbf{CP}^1$, E or $\Sigma_g, (g>1)$. This corresponds to the manifold being elliptic, parabolic or hyperbolic. As is usual, uniqueness will be in the sense of biregular, birational or unirational. In the non-compact case, one basically tries to tame infinity and compactify M as a Zariski open set of some projective algebraic variety $\bar{M}$ so that $M = \bar{M}\backslash\{\text{subvariety}\}$.

A. *Elliptic manifolds*

Frankel [Fr] conjectured that any compact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbf{CP}^n$; he proved this when $n = 2$. Later, Mori [Mo1] and Siu-Yau [SY1] proved the general case independently. In fact, Mori proved the Hartshorne conjecture under the weaker assumption that M has an ample tangent bundle.

The following is conjectured in [Y6]. If M is a simply connected compact Kähler manifold with nonnegative bisectional curvature, then M is isometric to a product of Hermitian symmetric spaces and complex projective spaces (not necessarily with Fubini-Study metric).

S. Bando [B1] proved this when $n = 3$. Mok and Zhong [MZ] proved that if, in addition, M is Einstein then M is biholomorphically isometric to a Hermitian symmetric space.

Recently, H.-D. Cao and B. Chow [CC] proved the conjecture assuming in addition M has nonnegative curvature operator. Even more recently, Mok claimed to prove the complete conjecture.

Let M^n be a compact Kähler manifold with positive Ricci curvature (this equivalent to $c_1(M) > 0$). We have the following questions:

(1) Under what condition is M^n unirational? Namely, does there exist a rational map from $\mathbf{CP}^n$ to M^n?

(2) Are there only a finite number (in the topological sense) of n-dimensional algebraic manifolds with positive first Chern class?

(3) Is it true that $c_1(M)^n$ is bounded by a constant depending only on n?

For $n = 2$, M^2 is a del Pezzo surface and (1), (2) and (3) are true. For $n = 3$. M^3 is a Fano 3-fold, i.e., an algebraic 3-manifold with ample anti-canonical bundle. Mori and Mukai [MM] give a complete classification of Fano 3-folds with second Betti number $b_2(M) \geqslant 2$. In fact, they proved that there are exactly 87 types of Fano 3-folds with $b_2(M) \geqslant 2$, up to deformation; moreover, a Fano 3-fold with $6 \leqslant b_2(M) \leqslant 10$ is isomorphic to $\mathbf{CP}^1 \times S_{11-b_2(M)}$ where S_d denotes the del Pezzo surface of degree d. The Fano 3-folds with $b_2 = 1$ are called Fano 3-folds of the first kind and were classified by Isokovskih [Is]. Using the above classification, questions (2) and (3) are easily checked to be true, but question (1) is not completely known even for $n = 3$. Using certain properties of conic fiber spaces over $\mathbf{CP}^2$, one can prove that some types of Feno's 3-folds, such as cubic 3-folds in $\mathbf{CP}^4$ are unirational. One does not know if every quartic 3-fold in $\mathbf{CP}^4$ is unirational; see the survey by Beauville [Be] for further details. By the way, before the classification of Mori and Mukai, S. M. L'vovskiĭ [Lv] proved that $c_1(M)^3 \leqslant c_1(\mathbf{CP}^3) = 64$ for Fano 3-folds by Riemann-Roch theorem and a detailed study of families of rational curves C with $(-K_M, C) = 4$. It is interesting to study the families of rational curves in Fano manifolds. Finally, for $n \geqslant 4$, the validity of (1), (2) and (3) are not known. Mori-Mukai recently proved M is uniruled. One more problem is if M^n is rationally connected. It is not hard to see that rational connectedness is stronger than uniruledness, but weaker than unirationalness.

Recall that Gromov's theorem [Gr] says that there is a constant $c(n)$ depending only on n such that $\sum_{i=0}^{n} b_i(M^n) \leqslant c(n)$ for any Riemannian manifold M^n with nonnegative sectional curvature. When M is Kähler, can one replace the condition "nonnegative sectional curvature" by "positive Ricci curvature"? One would also like to understand algebraic manifolds with Kodaira dimension $K(M) = -\infty$, i.e., $H^0(M, K^m) = 0$ for each $m > 0$, where K denotes the canonical line bundle. When $n = 2$, they are either rational surfaces or ruled surfaces.

B. Parabolic manifolds

Suppose M^n is a compact Kähler manifold which can be holomorphically covered by $\mathbf{C}^n$. Is it true that M^n can be also covered by the complex torus $T^n_{\mathbf{C}}$? For $n = 2$, Iitaka [Ii] proved that this is true. When $n \geqslant 3$, it is not known. Even in the case $n = 2$, the Kähler condition cannot be dropped (otherwise there exist counterexamples).

Let M^n be a noncompact complete Kähler manifold with positive sectional curvature; is M biholomorphic to $\mathbf{C}^n$? This question has been open for a long time. Siu-Yau [SY2] and Mok-Siu-Yau [MSY] proved the following. Let M be a complete noncompact Kähler manifold, $p \in M$ and $r(x) = \operatorname{dist}(x, p)$. Then

(a) If $\pi_1(M) = 0$, $-A/r^{2+\varepsilon} \leqslant K_M \leqslant 0$ for some $\varepsilon > 0$, then M is biholomorphic isometric to $\mathbf{C}^n$.

(b) If $|K_M| \leqslant A(1/r^2)^{1+\varepsilon}$ and A small enough, then M^n is biholomorphic to $\mathbf{C}^n$. If in addition $K_M < 0$, then M^n is isometric to $\mathbf{C}^n$ with the flat metric.

(c) If $K_M \geqslant 0$, $0 \leqslant R \leqslant A/r^{2+\varepsilon}$ and $\operatorname{vol}(B(p, r)) \geqslant Cr^{2n}$, then M is biholomorphic to $\mathbf{C}^n$.

Here A and C are any positive constants; K_M and R denote the sectional and scalar curvatures of M, respectively.

Mok [Mk1] improved these results by weakening the bound $1/r^{2+\varepsilon}$ to $1/r^2$. More precisely, he proved the following:

(d) If M has positive bisectional curvature, $0 < R \leqslant A/r^2$ and $\operatorname{vol}(B(p, r)) \geqslant Cr^{2n}$ for some positive constants A and C, then M is biholomorphic to an affine algebraic variety X.

Let M^n be an algebraic manifold with Kodaira dimension $K(M) = 0$, i.e., there exists $m_0 > 0$ such that $\dim H^0(M, K^{m_0}) > 0$, and for all $m \geqslant 0$, $\dim H^0(M, K^m) \leqslant C$ for some C independent of m, where K denotes the canonical line bundle. Can one classify these manifolds? Note that $c_1(M) = 0$ is a special case of $K(M) = 0$. When $n = 2$, there are exactly two classes of algebraic manifolds with Kodaira dimension $K(M) = 0$, quotients of abelian varieties or $K-3$ surfaces. For $n \geqslant 3$, this is unknown; the case $n = 3$ would be important for physics in view of the superstring theory. It is not known how to classify the topology of threefolds with $c_1 = 0$. Are there only finite number of such manifolds? Do they always admit rational curves if $\pi_1(M) = 0$?

C. Hyperbolic manifolds

If M^n is an algebraic manifold with negative sectional curvature, can M be holomorphically (branched) covered by a bounded domain $\Omega \subseteq \mathbf{C}^n$?

A weaker question is: If $\tilde{M}$ is a simply connected Kähler manifold with negative sectional curvature, are there enough bounded holomorphic functions on $\tilde{M}$ to separate points and give local coordinates? So far, no non-constant holomorphic functions have been proved to exist on $\tilde{M}$ even under the assumption that $\tilde{M}$ covers a compact manifold M.

B. Wong [Wo2] proved that if $\Omega \subseteq \mathbf{C}^n$ is a bounded domain with smooth boundary and Ω covers a compact manifold, then Ω is the ball. P. Yang [Yg] proved that if Ω is a bounded symmetric domain in $\mathbf{C}^n$ with rank greater than one, then there does not exist any Kähler metric on Ω with holomorphic bisectional curvature bounded between two negative constants. In particular, Ω cannot cover any compact Kähler manifold with negative bisectional curvature. Hence if a bounded domain Ω covers a compact Kähler manifold with negative curvature, it must be rather nonsmooth.

Recently, Mostow and Siu [MS] constructed a Kähler surface M^2 with negative sectional curvature by delicately piecing together the Poincaré metric of the 2-ball with the Bergman metric of the domain $\{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\}$ in $\mathbf{C}^n$. They proved that the universal cover $\tilde{M}$ of M is not the ball by showing that the Chern numbers of $\tilde{M}$ satisfy $c_1^2 < 3c_2$. This manifold is not diffeomorphic to a locally symmetric space and it is not known whether the universal cover is a bounded domain. Is it possible that a complete non-compact Kähler manifold with (topologically) trivial tangent bundle which covers a compact algebraic manifold is in fact biholomorphic to a domain?

For algebraic surfaces with positive canonical line bundle, does $|c_2/c_1^2 - 1/3|$ small enough imply that M has a Kähler metric with negative sectional curvature? This is not known.

The topology of algebraic surfaces is a very important subject. By the recent activity of Freedman and Donaldson, it seems reasonable to believe that every simply connected four-dimensional smooth manifold can be written as a connected sum of algebraic surfaces (possibly with different orientation). Very strong conclusions on the irreducibility of simply connected algebraic surfaces was recently asserted by Donaldson. Apparently only $\mathbf{CP}^2$ factors can occur if one wants to write it as a connected sum of differentiable manifolds. Perhaps simply connected four-dimensional manifolds with such irreducible condition is diffeomorphic to an algebraic surface.

It is more difficult to predict the topology of algebraic surfaces when the fundamental group is not finite. Shafarevich did make the conjecture that universal cover of any algebraic manifold is holomorphically convex. This may give some information about the topology besides the known inequality on Chern numbers.

4. Analytic objects

In order to understand the complex structure, it is important to understand the analytic objects attached to the structure. Here we give two examples:

A. *Holomorphic maps and vector bundles*

For a complex manifold M, the natural holomorphic vector bundles associated to it are TM, TM^*, $\Lambda^k TM$, $\otimes^k TM$, etc. Of special importance is the canonical line bundle $K = \Lambda^n TM^*$.

By blowing up points or submanifolds, one can get additional analytic objects. The Riemann-Roch theorem, which relates a topological invariant to an analytic invariant, is an important tool in constructing analytic objects or invariants from the given topological or analytic information.

The Yang-Mills theory is often useful in constructing holomorphic vector bundles and other objects over Kähler manifolds. Taubes [T1] used the anti-self-dual solutions to the Yang-Mills equations to construct holomorphic vector bundles of rank two over Kähler surfaces M^2. Is it possible to use this theory to recover the author's theorem that if M^2 is simply connected and its cup product is positive definite, then M^2 is biholomorphic to $\mathbf{CP}^2$?

Taubes [T2] also constructed holomorphic vector bundles over Kähler surfaces under the assumption of an inequality between the two Chern numbers (see also Donaldson [D1] and [D2]). So far, the above arguments only work in the two dimensional case. For higher dimensions, there is no good way to construct holomorphic vector bundles. The idea of Taubes can be extended to construct holomorphic vector bundles over high dimensional manifold. But it is not clear how large a class can one achieve in such a way.

B. *Analytic cycles*

Recall that by an analytic cycle, one simply means the formal sum of analytic subvarieties. Let M^n be an algebraic manifold and $V \subseteq M$ an analytic subvariety of codimension p. Then the fundamental cohomology class η_V of V belongs to $H^{p,p}(M) \cap H^{2p}(M; \mathbf{Z})$. Recall that an element