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Autor: Yau, Shing Tung
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then the limiting S^2 will enclose a fake disk. Take a Jordan curve on this S^2 so that it decomposes the S^2 into two regions with equal area. Then one expects this Jordan curve to bound an embedded minimal disk in the fake disk. If one can achieve this, one can shrink the S^2 more and obtain a contradiction which will give a proof of the Poincaré conjecture.

In conclusion, minimal surface theory is surprisingly successful in being applied to three dimensional topology. I believe that a more thorough study of minimal surfaces will reveal more secrets about three manifolds.

§ 5. Kähler Geometry

In the following we consider four basic topics in complex geometry.

1. Existence of complex and almost complex structure.
2. Existence of Kähler and algebraic structures on complex manifolds.
3. Uniformization problems and the parametrization of metrics.
4. Analytic objects over complex manifolds, e.g., analytic cycles, holomorphic vector bundles, etc.

We will divide this section into four parts corresponding to these topics.

1. Complex and almost complex structures

Let M be an even dimensional oriented differentiable manifold. The existence of an almost complex structure J is equivalent to a reduction of the structure group of the tangent bundle from $GL(2n, \mathbf{R})$ to $GL(n, \mathbf{C})$. This is basically an algebraic problem and is well understood.

However, the question of when an almost complex structure is homotopic to an integrable almost complex structure (i.e., one which comes from a complex structure) is much harder. When $n = 1$, every M^2 admits an almost complex structure and every such structure is integrable and algebraic. For $n = 2$, ven de Van [V1] gave several examples of compact M^4's which admit an almost complex structure but not a complex structure. His argument is based on the computations of the first and second Chern classes. When $n \geqslant 3$, there are no such examples known so far. In particular, we do not know whether or not the almost complex manifold S^6 admits a complex structure. This problem has been open for a long time.

The topology of complex surfaces is not well understood. By the works of Donaldson, one may believe that every simply connected four dimensional compact manifold is the connected sum of algebraic surfaces. For nonsimply connected algebraic surfaces, it is more difficult to speculate. The basic problem is to find a way to construct complex structures. Perhaps one can ask the following question. Suppose M is a compact almost complex manifold satisfying $\chi(M) = 3\tau(M)$ and covered topologically by $\mathbf{R}^4$. (Here $\chi(M)$ is the Euler number and $\tau(M)$ is the index of M.) If every abelian subgroup of $\pi_1(M)$ is infinite cyclic, does M admit a complex structure so that M is covered holomorphically by the unit ball in $\mathbf{C}^2$? The Lefschetz theorem may be useful in the above question.

2. Kähler and algebraic structures

Let M^n be an n complex dimensional compact manifold with complex structure J. The first question is: When is J Kählerian, i.e., (M, J) admits a Kähler metric? Harvey-Lawson [H-L] gave an intrinsic characterization of the Kählerian condition if and only if M carries no positive currents which are the (1, 1)-components of boundaries. Hodge theory gives a lot of necessary conditions for complex manifolds to be Kähler. In particular, their even Betti numbers must be positive and their odd Betti numbers are even. Also, when (M, J) is Kählerian, its rational homotopy type is determined by its rational cohomology, see Deligne-Griffiths-Morgan-Sullivan [DGMS].

Now suppose M is a Kähler manifold, i.e., M has some Kählerian complex structure. When does M admit a non-Kählerian complex structure? When does M have a unique complex (or Kählerian) structure?

When $n = 2$, every compact complex surface with even first Betti number is Kählerian. (This follows from the classification of Kodaira because Miyaoka [M1] and Siu [S1] proved respectively that elliptic surfaces with even first Betti number and $K-3$ surfaces are Kählerian. From this one concludes that among the seven classes of surfaces in Kodaira's classification, the first five are Kählerian for every complex structure. The remaining two classes of surfaces have odd first Betti number and hence admit no Kähler metrics. In particular, one sees that on a Kähler surface M^2, all complex structures on M^2 are Kählerian.)

When $n \geqslant 3$, the situation is much more complicated. Calabi [Ca3] proved that there is a non-Kählerian structure on $X \times T^2_{\mathbf{C}}$, where X is a hyperelliptic curve with genus $g = 2k + 1$, $k \geqslant 0$. On the other hand, we know that the only Kählerian structures on $X \times T^2_{\mathbf{C}}$ is the standard one.