

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 33 (1987)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** RADICALS AND HILBERT NULLSTELLENSATZ FOR NOT NECESSARILY ALGEBRAICALLY CLOSED FIELDS  
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**Kapitel:** §5. TWO EXAMPLES  
**DOI:** <https://doi.org/10.5169/seals-87902>

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In the case that  $k$  is an ordered field results similar to the Hilbert  $K$ -Nullstellensatz were proved by Dubois [2] and Risler [7]. To state their results we introduce the following notation:

Assume that  $k$  is an ordered field. Given an ideal  $I$  in  $R$  we let

$$I_D = \{f \in R \mid \text{there exists an integer } m, \text{ positive elements } a_1, a_2, \dots, a_m \text{ of } k \text{ and rational functions } u_1, u_2, \dots, u_m \text{ in } k(x_1, x_2, \dots, x_r) \text{ such that } f^n(1 + \sum_{i=1}^m a_i u_i^2) \in I\} \text{ and}$$

$$I_R = \{f \in R \mid \text{there are positive elements } a_2, a_3, \dots, a_m \text{ of } k \text{ and elements } f_2, f_3, \dots, f_m \text{ of } R \text{ such that } f^2 + \sum_{i=2}^m a_i f_i^2 \in I\}.$$

It is fairly easy to see that  $I_R$  and  $I_D$  are radical ideals and clearly  $I_R \subseteq I_D$ . The Hilbert Nullstellensatz of Risler [7] states that, if  $k = K = R_k$  where we denote by  $R_k$  the real closure of  $k$ , then

$$I_R = \{f \in R \mid Z_K(f) \supseteq Z_K(I)\}$$

and the Nullstellensatz of Dubois [2] that, if  $K = R_k$ , then

$$I_D = \{f \in R \mid Z_K(f) \supseteq Z_K(I)\}.$$

In particular it follows from these results that in the above cases  $I_R$  or  $I_D$  are equal to the  $K$ -radical  $\sqrt[K]{I}$ . From our point of view it is more satisfactory to proceed in the opposite direction and first prove directly, in the above cases, that the ideals  $I_D$  or  $I_R$  are equal to the  $K$ -radical and thus obtain the results of Dubois and Risler as a consequence of our  $K$ -Nullstellensatz. This can be done, however in order to prove that the various ideals are equal we need to use S. Lang's [6] version of Hilbert Nullstellensatz for real closed fields or Artin's solution of Hilbert's 17th problem (see [6], § 3 in particular Theorem 5 and Corollary 2 p. 279), so that this procedure is too close to the methods of Dubois and Risler to merit a separate presentation here.

## § 5. TWO EXAMPLES

In the introduction we associated to each ideal  $I$  of  $R$  a subset  $I_T$  of  $R$  such that  $I \subseteq I_T \subseteq \sqrt[K]{I}$ . For the two pairs of fields  $k = K = \mathbf{Z}/2\mathbf{Z} = GF(2)$  and  $k = K = \mathbf{Q}$  we give, in this section, examples of ideals  $I$  such that we have a strict inclusion  $I_T \subset \sqrt[K]{I}$ .

*Example 1.* Let  $k$  be the field with two elements and let  $K = k$ . Consider the ideal  $I = (x_1) \subseteq k[x_1, x_2] = R$ . The following three assertions hold:

(i) We have that

$$Z_K(I) = \{(0, 0), (0, 1)\} \subseteq \mathbf{A}_K^2 \text{ and } \{f \in R \mid f \text{ vanishes on } Z_K(I)\} = (x_1, x_2(x_2 + 1)).$$

(ii)  $\sqrt[K]{I} = (x_1, x_2(x_2 + 1))$ .

(iii)  $I_T = (x_1) = I$ .

In particular we have a strict inequality  $I_T \subset \sqrt[K]{I}$ .

Of the three assertions (i) is obvious and the second follows from (i) and the Hilbert  $K$ -Nullstellensatz. To prove assertion (iii) we let  $p \in P_K^0(m)$  and  $f_1, f_2, \dots, f_m$  be elements in  $R$  such that  $p(f_1, f_2, \dots, f_m) \in I$ . We shall prove that  $f_i \in I$  for  $i = 1, 2, \dots, m$ . Assume to the contrary that not all the  $f_i$  are in  $I$ . Then the polynomials  $f_i(0, x_2)$  are not all identically zero. Let  $d$  be the non-negative integer such that

$$f_i(0, x_2) = x_2^d g_i(x_2) \quad \text{for } i = 1, 2, \dots, m$$

and  $x_2$  does not divide  $g_j(x_2)$  for some index  $j$ . Since  $p(f_1, f_2, \dots, f_m) \in I$  we have that

$$p(f_1(0, x_2), f_2(0, x_2), \dots, f_m(0, x_2)) = x_2^{de} p(g_1(x_2), g_2(x_2), \dots, g_m(x_2))$$

is identically zero in  $k[x_2]$ , where  $e$  is the degree of  $p$ . Hence

$$p(g_1(x_2), g_2(x_2), \dots, g_m(x_2))$$

is identically zero. In particular we have that  $(g_1(0), g_2(0), \dots, g_m(0))$  is a zero of  $p$  in  $\mathbf{A}_K^m$  with  $g_j(0) \neq 0$ . This contradicts the assumption that  $p \in P_K^0(m)$ .

*Example 2.* Let  $k = K = \mathbf{Q}$  and let  $R = k[x_1, x_2, x_3]$ . Moreover, let

$$f(y_1, y_2, y_3) = y_1^3 + y_2^3 + 3y_3^3$$

and  $I = (f(y_1, y_2, y_3))$  the ideal in  $R$  generated by  $f$ .

The following three assertions hold:

(i) We have that  $Z_K(I) = \{(a, -a, 0) \mid a \in K\} \subseteq \mathbf{A}_K^3$  and

$$\{f \in R \mid f \text{ vanishes on } Z_K(I)\} = (x_1 + x_2, x_3).$$

(ii)  $\sqrt[K]{I} = (x_1 + x_2, x_3)$ .

(iii) The ideal  $I_T$  does not contain a (non-zero) linear form.

In particular we have a strict inequality  $I_T \subset \sqrt[K]{I}$ .

The first assertion of (i) is a well known result in number theory (see e.g. Hardy and Wright [3], Theorem 232 page 196) and the second assertion of (i) is an immediate consequence of the first. Assertion (ii) follows from (i) and the Hilbert  $K$ -Nullstellensatz.

To prove assertion (iii) we let  $l = ax_1 + bx_2 + cx_3$  be a non-zero linear form and  $p = p(y_1, y_2, \dots, y_m) \in P_K^0(m)$  an element of degree  $d$ . Assume that there are polynomials  $f_i = f_i(x_1, x_2, x_3)$  of  $R$  for  $i = 1, 2, \dots, m-1$  such that

$$p(f_1, f_2, \dots, f_{m-1}, l) = f(x_1, x_2, x_3) g(x_1, x_2, x_3)$$

for some polynomial  $g = g(x_1, x_2, x_3)$ . Then the following six assertions hold:

(a) *The polynomials  $f_1, f_2, \dots, f_{m-1}$  have zero constant term.*

Indeed, specialize  $x_1, x_2, x_3$  to 0, 0, 0 respectively. We obtain that

$$p(f_1(0, 0, 0), f_2(0, 0, 0), \dots, f_{m-1}(0, 0, 0), 0) = f(0, 0, 0) g(0, 0, 0) = 0.$$

Hence the existence of a non-zero constant term would contradict the assumption that  $p \in P_K^0(m)$ .

Denote by  $l_i = l_i(x_1, x_2, x_3)$  the linear term of  $f_i$ .

(b) *The homogenous polynomial  $p(l_1, l_2, \dots, l_{m-1}, l)$  is not (identically) zero and it is the lowest non-zero homogeneous term of*

$$p(f_1, f_2, \dots, f_{m-1}, l).$$

Indeed, if  $p(l_1, l_2, \dots, l_{m-1}, l)$  were zero, we can specialize  $(x_1, x_2, x_3)$  to a point  $(a_1, b_1, c_1)$  of  $K^3$  which is not a zero of  $l$ . We then obtain  $p(l_1(a_1, b_1, c_1), l_2(a_1, b_1, c_1), \dots, l_{m-1}(a_1, b_1, c_1), l(a_1, b_1, c_1)) = 0$  which again contradicts the assumption that  $p \in P_K^0(m)$ . The second assertion of (b) follows from (a).

Denote by  $h(x_1, x_2, x_3)$  the non-zero homogenous term of  $g(x_1, x_2, x_3)$  which has lowest degree.

(c) *We have that  $h(x_1, x_2, x_3)$  is of degree  $d-3$  and that*

$$p(l_1, l_2, \dots, l_{m-1}, l) = f(x_1, x_2, x_3) h(x_1, x_2, x_3).$$

Indeed, since  $f$  is homogeneous of degree 3, assertion (c) follows from assertion (b).

We write  $l_i = a_i x_1 + b_i x_2 + c_i x_3$  for  $i = 1, 2, \dots, m-1$ .

(d) *We have that  $a = b$  and that  $a_i = b_i$  for  $i = 1, 2, \dots, m-1$ .*

Indeed, specialize  $x_1, x_2, x_3$  to  $1, -1, 0$  respectively. From assertion (c) we obtain that

$$p(a_1 - b_1, a_2 - b_2, \dots, a_{m-1} - b_{m-1}, a - b) = f(1, -1, 0) h(1, -1, 0) = 0.$$

Hence assertion (d) follows from the assumption that  $p \in P_K^0(m)$ .

(e) *We have that  $a = b = a_i = b_i = 0$  for  $i = 1, 2, \dots, m - 1$ .*

Indeed, specializing  $x_1, x_2, x_3$  to  $x_1, x_2, 0$  respectively, we obtain from the equation of assertion (c) and from assertion (d) that

$$\begin{aligned} p(a_1(x_1 + x_2), a_2(x_1 + x_2), \dots, a_{m-1}(x_1 + x_2), a(x_1 + x_2)) \\ = (x_1^3 + x_2^3) h(x_1, x_2, 0). \end{aligned}$$

The left hand side of the latter equation is equal to

$$(x_1 + x_2)^d p(a_1, a_2, \dots, a_{m-1}, a)$$

which is not divisible by  $x_1^3 + x_2^3$  unless  $p(a_1, a_2, \dots, a_{m-1}, a) = 0$ . Assertion (e) therefore follows from assertion (d) and the assumption that  $p \in P_K^0(m)$ .

(f) *We have that  $c \neq 0$  and  $p(c_1, c_2, \dots, c_{m-1}, c) = 0$ .*

Indeed, since  $l = ax_1 + bx_2 + cx_3$  is non-zero it follows from assertion (e) that  $c \neq 0$ . Moreover it follows from assertion (e) that the equation of assertion (c) can be written as

$$p(c_1x_3, c_2x_3, \dots, c_{m-1}x_3, cx_3) = f(x_1, x_2, x_3) h(x_1, x_2, x_3).$$

The left hand side of the latter equation is equal to  $x_3^d p(c_1, c_2, \dots, c_{m-1}, c)$  which is not divisible by  $f(x_1, x_2, x_3)$  unless  $p(c_1, c_2, \dots, c_{m-1}, c) = 0$ .

We have thus proved that, if we assume that polynomials  $f_1, f_2, \dots, f_{m-1}$  such that  $p(f_1, f_2, \dots, f_{m-1}, l) \in I$  exist, we arrive at the contradiction (f) to the assumption that  $p \in P_K^0(m)$ . Hence we must have that  $l \notin I_T$  as asserted.