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To prove that, if  $K = \bar{k}$  and  $I$  is a proper ideal of  $R$ , we have that  $Z_K(I) \neq \emptyset$ , we choose a maximal ideal  $M$  containing  $I$ . By repeated application of assertion (ii) of Proposition 7 we see that there is a  $k$ -homomorphism

$$a: R/M \rightarrow \bar{k} = K$$

Hence, if  $\alpha_1, \alpha_2, \dots, \alpha_r$  are the classes of  $x_1, x_2, \dots, x_r$  in  $R/M$  we have that  $(a(\alpha_1), a(\alpha_2), \dots, a(\alpha_r)) \in Z_K(M) \cong Z_K(I)$  and  $Z_K(I) \neq \emptyset$  as we wanted to prove.

#### § 4. CONNECTIONS WITH PREVIOUS RESULTS

A less elegant form of the Hilbert  $K$ -Nullstellensatz, that do not involve the  $K$ -radical explicitly, is the following:

*Let  $J$  be an ideal of  $R$ . The following two assertions are equivalent:*

- (i) *If  $f \in R$  vanishes on  $Z_K(J)$ , then  $f \in J$ .*
- (ii) *If  $f_1, f_2, \dots, f_m$  are polynomials in  $R$  such that  $p(f_1, f_2, \dots, f_m) \in J$  for some  $p$  in  $P_K(m)$ , then  $f_m \in J$ .*

From Proposition 4 (ii) it follows that assertion (i) can be stated as

$$J = \{f \in R \mid Z_K(f) \cong Z_K(J)\}$$

and from the definition of the  $K$ -radical assertion (ii) can be stated as  $J = \sqrt[K]{J}$ . Hence the equivalence of the two assertions is the Hilbert  $K$ -Nullstellensatz for  $K$ -radical ideals. However, if  $I$  is any ideal of  $R$ , we have that  $J = \sqrt[K]{I}$  is  $K$ -radical by Proposition 3 and that  $Z_K(I) = Z_K(J)$  by Proposition 4 (i). Hence, the above result is equivalent to the Hilbert  $K$ -Nullstellensatz

$$\sqrt[K]{I} = \{f \in R \mid Z_K(f) \cong Z_K(I)\}$$

for  $I$ .

The sets  $P_K(m)$  in the particular case  $k = K$ , were introduced by Adkins, Gianni and Tognoli [1] in order to prove the above result when  $k = K$ . As a consequence they obtained the Hilbert Nullstellensatz in the particular case  $k = K = \bar{k}$ . The reason for introducing the sets  $P_K(m)$  in general is to formulate the above more general result, that is a true generalization of the Hilbert Nullstellensatz.

In the case that  $k$  is an ordered field results similar to the Hilbert  $K$ -Nullstellensatz were proved by Dubois [2] and Risler [7]. To state their results we introduce the following notation:

Assume that  $k$  is an ordered field. Given an ideal  $I$  in  $R$  we let

$$I_D = \left\{ f \in R \mid \text{there exists an integer } m, \text{ positive elements } a_1, a_2, \dots, a_m \text{ of } k \text{ and rational functions } u_1, u_2, \dots, u_m \text{ in } k(x_1, x_2, \dots, x_r) \text{ such that } f^n \left( 1 + \sum_{i=1}^m a_i u_i^2 \right) \in I \right\} \text{ and}$$

$$I_R = \left\{ f \in R \mid \text{there are positive elements } a_2, a_3, \dots, a_m \text{ of } k \text{ and elements } f_2, f_3, \dots, f_m \text{ of } R \text{ such that } f^2 + \sum_{i=2}^m a_i f_i^2 \in I \right\}.$$

It is fairly easy to see that  $I_R$  and  $I_D$  are radical ideals and clearly  $I_R \subseteq I_D$ . The Hilbert Nullstellensatz of Risler [7] states that, if  $k = K = R_k$  where we denote by  $R_k$  the real closure of  $k$ , then

$$I_R = \{ f \in R \mid Z_K(f) \cong Z_K(I) \}$$

and the Nullstellensatz of Dubois [2] that, if  $K = R_k$ , then

$$I_D = \{ f \in R \mid Z_K(f) \cong Z_K(I) \}.$$

In particular it follows from these results that in the above cases  $I_R$  or  $I_D$  are equal to the  $K$ -radical  $\sqrt[K]{I}$ . From our point of view it is more satisfactory to proceed in the opposite direction and first prove directly, in the above cases, that the ideals  $I_D$  or  $I_R$  are equal to the  $K$ -radical and thus obtain the results of Dubois and Risler as a consequence of our  $K$ -Nullstellensatz. This can be done, however in order to prove that the various ideals are equal we need to use S. Lang's [6] version of Hilbert Nullstellensatz for real closed fields or Artin's solution of Hilbert's 17th problem (see [6], § 3 in particular Theorem 5 and Corollary 2 p. 279), so that this procedure is too close to the methods of Dubois and Risler to merit a separate presentation here.

## § 5. TWO EXAMPLES

In the introduction we associated to each ideal  $I$  of  $R$  a subset  $I_T$  of  $R$  such that  $I \subseteq I_T \subseteq \sqrt[K]{I}$ . For the two pairs of fields  $k = K = \mathbf{Z}/2\mathbf{Z} = GF(2)$  and  $k = K = \mathbf{Q}$  we give, in this section, examples of ideals  $I$  such that we have a strict inclusion  $I_T \subset \sqrt[K]{I}$ .