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Artikel: RADICALS AND HILBERT NULLSTELLENSATZ FOR NOT NECESSARILY ALGEBRAICALLY CLOSED FIELDS

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Kapitel: §4. Connections with previous results

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To prove that, if $K = \bar{k}$ and I is a proper ideal of R, we have that $Z_K(I) \neq \emptyset$, we choose a maximal ideal M containing I. By repeated application of assertion (ii) of Proposition 7 we see that there is a k-homomorphism

$$a: R/M \to \bar{k} = K$$

Hence, if $\alpha_1, \alpha_2, ..., \alpha_r$ are the classes of $x_1, x_2, ..., x_r$ in R/M we have that $(a(\alpha_1), a(\alpha_2), ..., a(\alpha_n)) \in Z_K(M) \subseteqq Z_K(I)$ and $Z_K(I) \neq \emptyset$ as we wanted to prove.

§ 4. Connections with previous results

A less elegant form of the Hilbert K-Nullstellensatz, that do not involve the K-radical explicitely, is the following:

Let J be an ideal of R. The following two assertions are equivalent:

(i) *If $f \in R$ vanishes on $Z_K(J)$, then $f \in J$.*

(ii) *If $f_1, f_2, ..., f_m$ are polynomials in R such that $p(f_1, f_2, ..., f_m) \in J$ for some p in $P_K(m)$, then $f_m \in J$.*

From Proposition 4 (ii) it follows that assertion (i) can be stated as

$$J = \{f \in R \mid Z_K(f) \supseteqq Z_K(J)\}$$

and from the definition of the K-radical assertion (ii) can be stated as $J = \sqrt[K]{J}$. Hence the equivalence of the two assertions is the Hilbert K-Nullstellensatz for K-radical ideals. However, if I is any ideal of R, we have that $J = \sqrt[K]{I}$ is K-radical by Proposition 3 and that $Z_K(I) = Z_K(J)$ by Proposition 4 (i). Hence, the above result is equivalent to the Hilbert K-Nullstellensatz

$$\sqrt[K]{I} = \{f \in R \mid Z_K(f) \supseteqq Z_K(I)\}$$

for I.

The sets $P_K(m)$ in the particular case $k = K$, were introduced by Adkins, Gianni and Tognoli [1] in order to prove the above result when $k = K$. As a consequence they obtained the Hilbert Nullstellensatz in the particular case $k = K = \bar{k}$. The reason for introducing the sets $P_K(m)$ in general is to formulate the above more general result, that is a true generalization of the Hilbert Nullstellensatz.

In the case that k is an ordered field results similar to the Hilbert K-Nullstellensatz were proved by Dubois [2] and Risler [7]. To state their results we introduce the following notation:

Assume that k is an ordered field. Given an ideal I in R we let

$I_D = \{f \in R \mid$ there exists an integer m, positive elements $a_1, a_2, \ldots, a_m$ of k and rational functions $u_1, u_2, \ldots, u_m$ in $k(x_1, x_2, \ldots, x_r)$ such that $f^n(1 + \sum_{i=1}^{m} a_i u_i^2) \in I\}$ and

$I_R = \{f \in R \mid$ there are positive elements $a_2, a_3, \ldots, a_m$ of k and elements $f_2, f_3, \ldots, f_m$ of R such that $f^2 + \sum_{i=1}^{m} a_i f_i^2 \in I\}$.

It is fairly easy to see that I_R and I_D are radical ideals and clearly $I_R \subseteqq I_D$. The Hilbert Nullstellensatz of Risler [7] states that, if $k = K = R_k$ where we denote by R_k the real closure of k, then

$$I_R = \{f \in R \mid Z_K(f) \supseteqq Z_K(I)\}$$

and the Nullstellensatz of Dubois [2] that, if $K = R_k$, then

$$I_D = \{f \in R \mid Z_K(f) \supseteqq Z_K(I)\}.$$

In particular it follows from these results that in the above cases I_R or I_D are equal to the K-radical $\sqrt[K]{I}$. From our point of view it is more satisfactory to proceed in the opposite direction and first prove directly, in the above cases, that the ideals I_D or I_R are equal to the K-radical and thus obtain the results of Dubois and Risler as a consequence of our K-Nullstellensatz. This can be done, however in order to prove that the various ideals are equal we need to use S. Lang's [6] version of Hilbert Nullstellensatz for real closed fields or Artin's solution of Hilbert's 17th problem (see [6], § 3 in particular Theorem 5 and Corollary 2 p. 279), so that this procedure is too close to the methods of Dubois and Risler to merit a separate presentation here.

§ 5. Two examples

In the introduction we associated to each ideal I of R a subset I_T of R such that $I \subseteq I_T \subseteq \sqrt[K]{I}$. For the two pairs of fields $k = K = \mathbf{Z}/2\mathbf{Z} = GF(2)$ and $k = K = \mathbf{Q}$ we give, in this section, examples of ideals I such that we have a strict inclusion $I_T \subset \sqrt[K]{I}$.