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For which pair of fields  $k \subseteq K$  do we have that  $I_T = \sqrt[k]{I}$  for all ideals  $I$  of  $R$ ?

It was long conjectured that equality holds for all pairs of fields (at least when the characteristics of  $k$  is zero). We shall however, in section 5, give examples showing that one may have strict inequality  $I_T \subset \sqrt[k]{I}$  for the two pairs  $k = K = \mathbf{Z}/2\mathbf{Z}$  and  $k = K = \mathbf{Q}$ .

Before we proceed (in § 3) to prove the Hilbert  $K$ -Nullstellensatz we shall in § 2 collect all the results that we need about the  $K$ -radicals and the polynomials  $P_K(m)$  in the next section.

## § 2. SOME PROPERTIES OF THE $K$ -RADICAL

We shall denote by  $S(m)$  the polynomial ring  $k[y_1, y_2, \dots, y_m]$ .

LEMMA 1. Let  $p \in P_K(m)$  and  $q \in P_K(n)$ . For each polynomial  $s = s(y_1, y_2, \dots, y_{m+n}) \in S(m+n)$  of degree one less than  $q$ , we have that,

$$r = p(y_1 \cdot s, y_2 \cdot s, \dots, y_{m-1} \cdot s, q(y_{m+1}, y_{m+2}, \dots, y_{m+n})) \in P_K(m+n).$$

*Proof.* It is clear that  $r$  is a homogeneous polynomial in  $S(m+n)$ . Let  $(a_1, a_2, \dots, a_{m+n}) \in \mathbf{A}_K^{m+n}$  be a zero of  $r$ . Since  $p \in P_K(m)$ , we have that  $q(a_{m+1}, a_{m+2}, \dots, a_{m+n}) = 0$ . However, we have that  $q \in P_K(m)$  so that  $a_{m+n} = 0$ . Consequently  $r \in P_K(m+n)$  as asserted.

PROPOSITION 2. Let  $A$  be a  $k$ -algebra and  $I$  an ideal of  $A$ . Then the  $K$ -radical  $\sqrt[k]{I}$  of  $I$  is an ideal of  $A$  (possibly  $A$  itself) which contains the radical of  $I$ .

*Proof.* Since  $P_K(1) = \{1, y_1, y_1^2, \dots\}$  it is clear that the set  $\sqrt[k]{I}$  contains  $\sqrt{I}$ .

Let  $f$  and  $g$  be elements in  $\sqrt[k]{I}$ . Then by the definition of the  $K$ -radical there are positive integers  $m$  and  $n$ , polynomials  $p \in P_K(m)$  and  $q \in P_K(n)$  and elements  $f_1, f_2, \dots, f_{m-1}$  and  $g_1, g_2, \dots, g_{n-1}$  of  $A$  such that

$$p(f_1, f_2, \dots, f_{m-1}) \in I \quad \text{and}$$

$$q(g_1, g_2, \dots, g_{n-1}, g) \in I$$

Let  $h$  be an element of  $A$  and let  $d$  be the degree of  $p$ . Then we have that

$$p(hf_1, hf_2, \dots, hf_{m-1}, hf) = h^d p(f_1, f_2, \dots, f_{m-1}, f) \in I.$$

Consequently it follows from the definition of  $K$ -radicals that  $h \cdot f \in \sqrt[d]{I}$ .

In order to prove the Proposition it remains to prove that  $(f + g) \in \sqrt[d]{I}$ .

To this end we rewrite the polynomial  $q(y_{m+1}, y_{m+2}, \dots, y_{m+n})$  in the following form

$$q(y_{m+1}, y_{m+2}, \dots, y_{m+n-1}, y_{m+n} - y_m) + y_m^s q(y_{m+1}, y_{m+2}, \dots, y_{m+n}),$$

where  $s$  is a homogeneous polynomial of  $S(m+n)$  of degree one less than the degree of  $q$ .

By Lemma 1 we have that

$$r = r(y_1, y_2, \dots, y_{m+n}) = p(y_1 \cdot s, y_2 \cdot s, \dots, y_{m-1} \cdot s, q(y_{m+1}, y_{m+2}, \dots, y_{m+n}))$$

is in  $P_K(m+n)$ . However, from the above form of  $q(y_{m+1}, y_{m+2}, \dots, y_{m+n})$ , it follows that  $r$  can be rewritten as

$$p(y_1 \cdot s, y_2 \cdot s, \dots, y_{m-1} \cdot s, y_m \cdot s) + q(y_{m+1}, y_{m+2}, \dots, y_{m+n-1}, y_{m+n} - y_m) \cdot t(y_1, y_2, \dots, y_{m+n}),$$

where  $t(y_1, y_2, \dots, y_{m+n})$  is a homogeneous polynomial in  $S(m+n)$  of degree equal to  $(d-1) \cdot \deg(q)$ .

From the latter form of  $r$  we obtain that, if we write  $l = s(f, g_1, g_2, \dots, g_n)$ , then  $h = r(f_1, f_2, \dots, f_{m-1}, f, g_1, g_2, \dots, g_{n-1}, f+g)$  can be written as

$$l^d p(f_1, f_2, \dots, f_{m-1}, f) + q(g_1, g_2, \dots, g_{n-1}, g) \cdot t(f_1, f_2, \dots, f_{m-1}, f, g_1, g_2, \dots, g_{n-1}, f+g).$$

The latter element is in  $I$  and since  $r \in P_K(m+n)$  it follows from the definition of the  $K$ -radical that  $f + g \in \sqrt[d]{I}$ , as we wanted to prove.

We shall call an ideal  $I$  of a  $k$ -algebra  $A$ ,  $K$ -radical, if  $\sqrt[d]{I} = I$ .

The next result shows that  $\sqrt[d]{I}$  is always  $K$ -radical.

**PROPOSITION 3.** *Let  $A$  be a  $k$ -algebra and  $I$  an ideal of  $A$ . Moreover, let  $J = \sqrt[d]{I}$ . Then we have that  $\sqrt[d]{J} = J$ .*

*Proof.* Let  $f$  be in  $\sqrt[d]{J}$ . We shall prove that  $f \in J$ . By definition of the  $K$ -radical, there is a positive integer  $n$ , a polynomial  $q \in P_K(n)$  and elements  $f_1, f_2, \dots, f_{n-1}$  in  $A$  such that

$$g = q(f_1, f_2, \dots, f_{n-1}, f) \in J.$$

Now, since  $g \in \sqrt[d]{I}$ , there is furthermore a positive integer  $m$ , a polynomial  $p \in P_K(m)$  and elements  $g_1, g_2, \dots, g_{m-1}$  in  $A$  such that

$$p(g_1, g_2, \dots, g_{m-1}, g) \in I.$$

Let  $d$  be the degree of  $q$ . Then by Lemma 1 with  $s = y_n^{d-1}$  we have that

$$\begin{aligned} & r(y_1, y_2, \dots, y_{m+n}) \\ &= p(y_1 \cdot y_m^{d-1}, y_2 \cdot y_m^{d-1}, \dots, y_{m-1} \cdot y_m^{d-1}, q(y_{m+1}, y_{m+2}, \dots, y_n)) \end{aligned}$$

is in  $P_K(m+n)$ . However, we have that the element

$$\begin{aligned} & r(g_1, g_2, \dots, g_{m-1}, 1, f_1, f_2, \dots, f_{n-1}, f) \\ &= p(g_1, g_2, \dots, g_{m-1}, q(f_1, f_2, \dots, f_{n-1}, f)) = p(g_1, g_2, \dots, g_{m-1}, g) \end{aligned}$$

is in  $I$ . Hence  $f$  is in  $\sqrt[K]{I} = J$  as we wanted to prove.

As in the traditional case, one of the two assertions of the Hilbert  $K$ -Nullstellensatz and of its weak form is easy.

**PROPOSITION 4.** *Let  $I$  be an ideal of  $R$  and  $J = \sqrt[K]{I}$ . Then the following assertions hold:*

- (i)  $Z_K(J) = Z_K(I)$ ,
- (ii)  $J \subseteq \{f \in R \mid Z_K(f) \supseteq Z_K(I)\}$ ,
- (iii) if  $Z_K(I) \neq \emptyset$  then  $J \neq R$ .

*Proof.* Since  $J$  contains  $I$  we have the inclusion  $Z_K(J) \subseteq Z_K(I)$ . To prove the opposite inclusion as well as assertion (ii) it suffices to prove that for each point  $a = (a_1, a_2, \dots, a_r) \in \mathbf{A}_K^r$  of  $Z_K(I)$ , we have that  $f(a) = 0$  for all  $f \in J$ . However if  $f \in J$  then there exists a polynomial  $p$  in  $P_K(m)$  for some natural number  $m$  and elements  $f_1, f_2, \dots, f_{m-1}$  in  $R$  such that

$$p(f_1, f_2, \dots, f_{m-1}, f) \in I.$$

Since  $a$  is in  $Z_K(I)$  we obtain that

$$p(f_1(a), f_2(a), \dots, f_{m-1}(a), f(a)) = 0.$$

However, we have that  $p \in P_K(m)$  so that  $f(a) = 0$ .

The last assertion of the Proposition follows from assertion (ii).

The crucial tool in our proof of the Hilbert  $K$ -Nullstellensatz is the following result, which certainly is well known, but for which we have no reference.

**PROPOSITION 5.** *Assume that  $K$  is not algebraically closed. Then, for each positive integer  $m$ , there is a homogeneous polynomial  $p \in k[y_1, y_2, \dots, y_m]$  with only the trivial zero in  $\mathbf{A}_K^m$ . That is,  $Z_K(p) = (0, 0, \dots, 0)$ .*

*Proof.* For  $m = 1$  we can use  $p(y_1) = y_1$ . The heart of the proof is the case  $m = 2$ . We divide the proof for  $m = 2$  into two cases.

*Case 1.* There exists an element  $\alpha$  in  $\bar{k} \setminus K$  which is separable over  $k$ . Let  $L$  be the normal closure of  $k(\alpha)$  in  $\bar{k}$ . Then  $L$  is a finite separable extension of  $k$  and thus generated by one element  $\beta$ . That is  $L = k(\beta)$ . Since  $L$  is normal all the conjugates  $\beta = \beta_1, \beta_2, \dots, \beta_n$  of  $\beta$  are in  $L$  and clearly  $L = k(\beta_i)$  for  $i = 1, 2, \dots, n$ . We have that  $L$  is not contained in  $K$  because  $\alpha \notin K$ . Hence, none of the roots  $\beta_1, \beta_2, \dots, \beta_n$  of the minimal polynomial  $f(x) \in k[x]$  of the element  $\beta$  over  $k$ , are in  $K$ . Consequently, the homogenization.

$$p(y_1, y_2) = y_2^d \cdot f(y_1 \cdot y_2^{-1})$$

of  $f$ , where  $d$  is the degree of  $f$ , has no non-trivial root in  $\mathbf{A}_K^2$ .

*Case 2.* All elements of  $\bar{k} \setminus K$  are purely inseparable over  $k$ . Choose an element  $\gamma \in \bar{k} \setminus K$ . Then  $\gamma^q = a$  is in  $k$  for some power  $q$  of the characteristics of  $k$  and  $\gamma$  is the only root of the polynomial  $x^q - a$ . Hence

$$p(y_1, y_2) = (y_1 - a y_2)^q$$

is a homogeneous polynomial without any non-trivial roots in  $\mathbf{A}_K^2$ .

The two cases above exhaust all possibilities for elements in  $\bar{k} \setminus K$ . Hence we have proved the existence of homogeneous polynomials in  $k[y_1, y_2]$  without any non trivial zeroes.

We now proceed by induction on  $m$ . Assume that  $m \geq 2$  and that we have proved the existence of a homogeneous polynomial  $p(y_1, y_2, \dots, y_m)$  with only the trivial zero in  $\mathbf{A}_K^m$ . Let  $q(y_1, y_2)$  be a homogeneous polynomial with only the trivial zero in  $\mathbf{A}_K^2$ . Then, if  $d$  is the degree of  $p$ , we have that  $r(y_1, y_2, \dots, y_{m+1}) = q(p(y_1, y_2, \dots, y_m), y_{m+1}^d)$  is a homogeneous polynomial with only the trivial zero in  $\mathbf{A}_K^{m+1}$ . Indeed, the homogeneity is clear, and if  $(a_1, a_2, \dots, a_{m+1}) \in \mathbf{A}_K^{m+1}$  is a zero of  $r$ , we must have that  $p(a_1, a_2, \dots, a_m) = 0$  and  $a_{m+1} = 0$  since  $q$  has no non-trivial zeroes. Then we must have that  $a_1 = a_2 = \dots = a_m = 0$  since the same is true for  $p$ .

### § 3. PROOF OF THE HILBERT $K$ -NULLSTELLENSATZ

There exists in the literature a great variety of proofs of the Hilbert Nullstellensatz. Most of them start by proving the weak form and then deducing the Nullstellensatz by localization procedures that are more or less