Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 33 (1987)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: RADICALS AND HILBERT NULLSTELLENSATZ FOR NOT NECESSARILY ALGEBRAICALLY CLOSED FIELDS

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Kapitel: §2. SOME PROPERTIES OF THE K-RADICAL

DOI: https://doi.org/10.5169/seals-87902

For which pair of fields $k \subseteq K$ *do we have that* $I_T = \sqrt[K]{I}$ *for all ideals* I *of* R?

It was long conjectured that equality holds for all pairs of fields (at least when the characteristics of k is zero). We shall however, in section 5, give examples showing that one may have strict inequality $I_T \subset \sqrt[K]{I}$ for the two pairs $k = K = \mathbf{Z}/2\mathbf{Z}$ and $k = K = \mathbf{Q}$.

Before we proceed (in § 3) to prove the Hilbert K-Nullstellensatz we shall in § 2 collect all the results that we need about the K-radicals and the polynomials $P_K(m)$ in the next section.

§ 2. Some properties of the K-radical

We shall denote by $S(m)$ the polynomial ring $k[y_1, y_2, ..., y_m]$.

Lemma 1. *Let* $p \in P_K(m)$ *and* $q \in P_k(n)$. *For each polynomial* $s = s(y_1, y_2, ..., y_{m+n}) \in S(m+n)$ *of degree one less than* q, *we have that,*

$$r = p(y_1 \cdot s, y_2 \cdot s, ..., y_{m-1} \cdot s, q(y_{m+1}, y_{m+2}, ..., y_{m+n})) \in P_K(m+n) .$$

Proof. It is clear that r is a homogeneous polynomial in $S(m+n)$. Let $(a_1, a_2, \ldots, a_{m+n}) \in \mathbf{A}_K^{m+n}$ be a zero of r. Since $p \in P_K(m)$, we have that $q(a_{m+1}, a_{m+2}, ..., a_{m+n}) = 0$. However, we have that $q \in P_K(m)$ so that $a_{m+n} = 0$. Consequently $r \in P_K(m+n)$ as asserted.

Proposition 2. *Let* A *be a k-algebra and* I *an ideal of* A. *Then the K-radical* $\sqrt[K]{I}$ *of* I *is an ideal of* A *(possibly* A *itself) which contains the radical of* I.

Proof. Since $P_K(1) = \{1, y_1, y_1^2, ...\}$ it is clear that the set $\sqrt[K]{I}$ contains $\sqrt{I}$.

Let f and g be elements in $\sqrt[K]{I}$. Then by the definition of the K-radical there are positive integers m and n, polynomials $p \in P_K(m)$ and $q \in P_K(n)$ and elements $f_1, f_2, ..., f_{m-1}$ and $g_1, g_2, ..., g_{n-1}$ of A such that

$$p(f_1, f_2, ..., f_{m-1}) \in I \quad \text{and}$$
$$q(g_1, g_2, ..., g_{n-1}, g) \in I$$

Let h be an element of A and let d be the degree of p. Then we have that

$$p(hf_1, hf_2, ..., hf_{m-1}, hf) = h^d p(f_1, f_2, ..., f_{m-1}, f) \in I .$$

Consequently it follows from the definition of K-radicals that $h \cdot f \in \sqrt[K]{I}$.

In order to prove the Proposition it remains to prove that $(f+g) \in \sqrt[K]{I}$.

To this end we rewrite the polynomial $q(y_{m+1}, y_{m+2}, \dots, y_{m+n})$ in the following form

$$q(y_{m+1}, y_{m+2}, \dots, y_{m+n-1}, y_{m+n}-y_m) + y_m s(y_m, y_{m+1}, \dots, y_{m+n}),$$

where s is a homogeneous polynomial of $S(m+n)$ of degree one less than the degree of q.

By Lemma 1 we have that

$$r = r(y_1, y_2, \dots, y_{m+n}) = p(y_1 \cdot s, y_2 \cdot s, \dots, y_{m-1} \cdot s, q(y_{m+1}, y_{m+2}, \dots y_{m+n}))$$

is in $P_K(m+n)$. However, from the above form of $q(y_{m+1}, y_{m+2}, \dots y_{m+n})$, it follows that r can be rewritten as

$$p(y_q \cdot s, y_2 \cdot s, \dots, y_{m-1} \cdot s, y_m \cdot s) + q(y_{m+1}, y_{m+2}, \dots, y_{m+n-1}, y_{m+n}-y_m) \\ \cdot t(y_1, y_2, \dots, y_{m+n}),$$

where $t(y_1, y_2, \dots, y_{m+n})$ is a homogeneous polynomial in $S(m+n)$ of degree equal to $(d-1) \cdot \deg(q)$.

From the latter form of r we obtain that, if we write $l = s(f, g_1, g_2, \dots, g_n)$, then $h = r(f_1, f_2, \dots, f_{m-1}, f, g_1, g_2, \dots, g_{n-1}, f+g)$ can be written as

$$l^d p(f_1, f_2, \dots, f_{m-1}, f) \\ + q(g_1, g_2, \dots, g_{n-1}, g) \cdot t(f_1, f_2, \dots, f_{m-1}, f, g_1, g_2, \dots, g_{n-1}, f+g).$$

The latter element is in I and since $r \in P_K(m+n)$ it follows from the definition of the K-radical that $f + g \in \sqrt[K]{I}$, as we wanted to prove.

We shall call an ideal I of a k-algebra A, K-radical, if $\sqrt[K]{I} = I$.

The next result shows that $\sqrt[K]{I}$ is always K-radical.

Proposition 3. *Let A be a k-algebra and I an ideal of A. Moreover, let $J = \sqrt[K]{I}$. Then we have that $\sqrt[K]{J} = J$.*

Proof. Let f be in $\sqrt[K]{J}$. We shall prove that $f \in J$. By definition of the K-radical, there is a positive integer n, a polynomial $q \in P_K(n)$ and elements $f_1, f_2, \dots f_{n-1}$ in A such that

$$g = q(f_1, f_2, \dots, f_{n-1}, f) \in J.$$

Now, since $g \in \sqrt[K]{I}$, there is furthermore a positive integer n, a polynomial $p \in P_K(m)$ and elements $g_1, g_2, \dots g_{m-1}$ in A such that

$$p(g_1, g_2, \ldots, g_{m-1}, g) \in I .$$

Let d be the degree of q. Then by Lemma 1 with $s = y_n^{d-1}$ we have that

$$r(y_1, y_2, \ldots, y_{m+n})$$
$$= p(y_1 \cdot y_m^{d-1}, y_2 \cdot y_m^{d-1}, \ldots, y_{m-1} \cdot y_m^{d-1}, q(y_{m+1}, y_{m+2}, \ldots, y_n))$$

is in $P_K(m+n)$. However, we have that the element

$$r(g_1, g_2, \ldots, g_{m-1}, 1, f_1, f_2, \ldots, f_{n-1}, f)$$
$$= p(g_1, g_2, \ldots, g_{m-1}, q(f_1, f_2, \ldots, f_{n-1}, f)) = p(g_1, g_2, \ldots, g_{m-1}, g)$$

is in I. Hence f is in $\sqrt[K]{I} = J$ as we wanted to prove.

As in the traditional case, one of the two assertions of the Hilbert K-Nullstellensatz and of its weak form is easy.

PROPOSITION 4. *Let* I *be an ideal of* R *and* $J = \sqrt[K]{I}$. *Then the following assertions hold:*

(i) $Z_K(J) = Z_K(I)$,

(ii) $J \subseteqq \{f \in R \mid Z_K(f) \supseteqq Z_K(I)\}$,

(iii) if $Z_K(I) \neq \emptyset$ then $J \neq R$.

Proof. Since J contains I we have the inclusion $Z_K(J) \subseteqq Z_K(I)$. To prove the opposite inclusion as well as assertion (ii) it suffices to prove that for each point $a = (a_1, a_2, \ldots, a_r) \in \mathbf{A}_K^r$ of $Z_K(I)$, we have that $f(a) = 0$ for all $f \in J$. However if $f \in T$ then there exists a polynomial p in $P_K(m)$ for some natural number m and elements $f_1, f_2, \ldots, f_{m-1}$ in R such that

$$p(f_1, f_2, \ldots, f_{m-1}, f) \in I .$$

Since a is in $Z_K(I)$ we obtain that

$$p(f_1(a), f_2(a), \ldots, f_{m-1}(a), f(a)) = 0 .$$

However, we have that $p \in P_K(m)$ so that $f(a) = 0$.

The last assertion of the Proposition follows from assertion (ii).

The crucial tool in our proof of the Hilbert K-Nullstellensatz is the following result, which certainly is well known, but for which we have no reference.

PROPOSITION 5. *Assume that* K *is not algebraically closed. Then, for each positive integer* m, *there is a homogeneous polynomial* $p \in k[y_1, y_2, \ldots, y_m]$ *with only the trivial zero in* $\mathbf{A}_K^m$. *That is,* $Z_K(p) = (0, 0, \ldots, 0)$.

Proof. For $m = 1$ we can use $p(y_1) = y_1$. The heart of the proof is the case $m = 2$. We divide the proof for $m = 2$ into two cases.

Case 1. There exists an element α in $\bar{k}\backslash K$ *which is separable over* k. Let L be the normal closure of $k(\alpha)$ in $\bar{k}$. Then L is a finite separable extension of k and thus generated by one element β. That is $L = k(\beta)$. Since L is normal all the conjugates $\beta = \beta_1, \beta_2, \dots, \beta_n$ of β are in L and clearly $L = k(\beta_i)$ for $i = 1, 2, \dots, n$. We have that L is not contained in K because $\alpha \notin K$. Hence, none of the roots $\beta_1, \beta_2, \dots, \beta_n$ of the minimal polynomial $f(x) \in k[x]$ of the element β over k, are in K. Consequently, the homogenization.

$$p(y_1, y_2) = y_2^d \cdot f(y_1 \cdot y_2^{-1})$$

of f, where d is the degree of f, has no non-trivial root in $\mathbf{A}_K^2$.

Case 2. All elements of $\bar{k}\backslash K$ *are purely inseparable over* k. Choose an element $\gamma \in \bar{k}\backslash K$. Then $\gamma^q = a$ is in k for some power q of the characteristics of k and γ is the only root of the polynomial $x^q - a$. Hence

$$p(y_1, y_2) = (y_1 - a\, y_2)^q$$

is a homogeneous polynomial without any non-trivial roots in $\mathbf{A}_K^2$.

The two cases above exhaust all possibilities for elements in $\bar{k}\backslash K$. Hence we have proved the existence of homogeneous polynomials in $k[y_1, y_2]$ without any non trivial zeroes.

We now proceed by induction on m. Assume that $m \geqslant 2$ and that we have proved the existence of a homogeneous polynomial $p(y_1, y_2, \dots, y_m)$ with only the trivial zero in $\mathbf{A}_K^m$. Let $q(y_1, y_2)$ be a homogeneous polynomial with only the trivial zero in $\mathbf{A}_K^2$. Then, if d is the degree of p, we have that $r(y_1, y_2, \dots, y_{m+1}) = q(p(y_1, y_2, \dots, y_m), y_{m+1}^d)$ is a homogeneous polynomial with only the trivial zero in $\mathbf{A}_K^{m+1}$. Indeed, the homogeneity is clear, and if $(a_1, a_2, \dots, a_{m+1}) \in \mathbf{A}_K^{m+1}$ is a zero of r, we must have that $p(a_1, a_2, \dots, a_m) = 0$ and $a_{m+1} = 0$ since q has no non-trivial zeroes. Then we must have that $a_1 = a_2 = \dots = a_m = 0$ since the same is true for p.

§ 3. Proof of the Hilbert K-Nullstellensatz

There exists in the literature a great variety of proofs of the Hilbert Nullstellensatz. Most of them start by proving the weak form and then deducing the Nullstellensatz by localization procedures that are more or less