Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 33 (1987)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AN ORBIT CLOSING PROOF OF BROUWER'S LEMMA ON TRANSLATION ARCS

Autor: Fathi, Albert

Kapitel: 2. Fixed points for homeomorphisms of the sphere

DOI: https://doi.org/10.5169/seals-87901

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 13.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

2. Fixed points for homeomorphisms of the sphere

The next lemma is the second important ingredient. It can be proved by Nielsen's theory of fixed points. We will give a direct proof.

LEMMA 2.1. *Let* $h: \mathbf{S}^2 \to \mathbf{S}^2$ *be an orientation preserving homeomorphism. If* h *has a period 2 point which is not a fixed point, then the set* $\mathrm{Fix}(h)$ *can be written as a disjoint union* $\mathrm{Fix}(h) = F_1 \cup F_2$ *with* F_1 *and* F_2 *closed non empty and having point index equal to 1.*

Proof. Call x the point of period 2. Remark that since h preserve the orientation it induces on $\pi_1(\mathbf{S}^2 \backslash \{x, h(x)\}) = \mathbf{Z}$ the map $x \mapsto -x$. Choose an essential annulus $A \subset \mathbf{S}^2 \backslash \{x, h(x)\}$ large enough so that when we compose $h: A \to \mathbf{S}^2 \backslash \{x, h(x)\}$ with a retraction of $\mathbf{S}^2 \backslash \{x, h(x)\}$ on A we obtain a map $\bar{h}: A \to A$ which has no fixed point on the boundary, has the same fixed point as h and is equal to h in a neighborhood of the set of fixed points $\mathrm{Fix}(h) = \mathrm{Fix}(\bar{h})$. We will call $\tilde{A} \to A$ the universal cover of A of course $\tilde{A} = [0, 1] \times \mathbf{R}$ and if we denote by T a generator of the group of deck transformation of $\tilde{A} \to A$, we can write under this identification $T(x) = x + 1$ where addition is to be taken in the $\mathbf{R}$ coordinate. The map $\bar{h}$ lifts to a proper map $\tilde{h}$ which verifies $\tilde{h}T = T^{-1}\tilde{h}$. It follows that $\tilde{h}$ can be extended to the compactification of $\tilde{A}$ by its two ends $\varepsilon_-, \varepsilon_+$ by a map which exchange these to ends. Since $\tilde{A} \cup \{\varepsilon_-, \varepsilon_+\}$ is homeomorphic to a disk $\tilde{h}$ has a non empty compact set $\tilde{F}_1$ of fixed points which does not intersect the boundary because $\tilde{h}$ exchange ε_- and ε_+ and $\bar{h}$ has no fixed point on the boundary of A. Remark that the index of $\tilde{F}_1$ is 1. Moreover, the map $\tilde{A} \to A$ is injective on $\tilde{F}_1$ because if $\tilde{h}(x) = x$ we have $\tilde{h}(x+n) = x - n \neq x + n$ if $n \neq 0$. Since $\tilde{A} \to A$ is a covering it is clear that this map is also injective in a neighborhood of $\tilde{F}_1$. It follows that the image F_1 of $\tilde{F}_1$ under $\tilde{A} \to A$ is a compact non empty set of fixed points of $\bar{h}$ which has index 1. If $x \in \tilde{F}_1$, we have $T\tilde{h}(x+n) = T(x-n) = x + 1 - n \neq x + n$ for all n because $1/2 \notin \mathbf{Z}$. It follows that F_2, the image under $\tilde{A} \to A$ of $\mathrm{Fix}(T\tilde{h})$—which is also a compact non empty set of fixed points of $\bar{h}$ with index 1—is disjoint from F_1. If $x \in A$ is a fixed point of $\bar{h}$, it lifts to a point $\tilde{x} \in \tilde{A}$ which verifies $\tilde{h}(\tilde{x}) = \tilde{x} + n$. If $n = 2k$ then $\tilde{h}(\tilde{x}+k) = \tilde{h}(\tilde{x}) - k = \tilde{x} + 2k - k = \tilde{x} + k$. If $n = 2k - 1$ then $T\tilde{h}(\tilde{x}+k) = T(\tilde{x}+2k-1-k) = \tilde{x} + k$. This shows clearly that $\mathrm{Fix}(\bar{h}) = F_1 \cup F_2$. Since $\bar{h}$ is equal to h in a neighborhood of $\mathrm{Fix}(\bar{h}) = \mathrm{Fix}(h)$, this ends the proof. □

If we combine the Main Lemma 1.1 and lemma 2.1, we obtain:

LEMMA 2.2. *Let* $h: \mathbf{S}^2 \to \mathbf{S}^2$ *be an orientation preserving homeomorphism. If* h *has a non wandering point which is not a fixed point, then the set* $\mathrm{Fix}(h)$ *can be written as a disjoint union* $\mathrm{Fix}(h) = F_1 \cup F_2$ *with* F_1 *and* F_2 *closed non empty and having fixed point index equal to 1.*

Since we can compactify an orientation preserving homeomorphism of $\mathbf{R}^2$ by an orientation preserving homeomorphism of $\mathbf{S}^2$ with one more fixed point at infinity, we obtain the next two corollaries.

COROLLARY 2.3. (Brouwer's Lemma on translation arcs). *Let* $h: \mathbf{R}^2 \to \mathbf{R}^2$ *be a fixed point free orientation preserving homeomorphism. Then* h *has no periodic point, each point wanders under* h. *Moreover, if* α *is a translation arc, the union* $\bigcup_{n \in \mathbf{Z}} h^n(\alpha)$ *is homeomorphic to a line and it does not accumulate on itself.*

COROLLARY 2.4. *Let* $h: \mathbf{R}^2 \to \mathbf{R}^2$ *be an orientation preserving homeomorphism. If the non wandering set of* h *is not reduced to the set of fixed points then there is a compact non empty subset* $F \subset \mathrm{Fix}(h)$ *which has fixed point index equal to 1.*