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Autor: Fathi, Albert
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2. FIXED POINTS FOR HOMEOMORPHISMS OF THE SPHERE

The next lemma is the second important ingredient. It can be proved by Nielsen's theory of fixed points. We will give a direct proof.

LEMMA 2.1. *Let $h: \mathbf{S}^2 \rightarrow \mathbf{S}^2$ be an orientation preserving homeomorphism. If h has a period 2 point which is not a fixed point, then the set $\text{Fix}(h)$ can be written as a disjoint union $\text{Fix}(h) = F_1 \cup F_2$ with F_1 and F_2 closed non empty and having point index equal to 1.*

Proof. Call x the point of period 2. Remark that since h preserve the orientation it induces on $\pi_1(\mathbf{S}^2 \setminus \{x, h(x)\}) = \mathbf{Z}$ the map $x \mapsto -x$. Choose an essential annulus $A \subset \mathbf{S}^2 \setminus \{x, h(x)\}$ large enough so that when we compose $h: A \rightarrow \mathbf{S}^2 \setminus \{x, h(x)\}$ with a retraction of $\mathbf{S}^2 \setminus \{x, h(x)\}$ on A we obtain a map $\bar{h}: A \rightarrow A$ which has no fixed point on the boundary, has the same fixed point as h and is equal to h in a neighborhood of the set of fixed points $\text{Fix}(h) = \text{Fix}(\bar{h})$. We will call $\tilde{A} \rightarrow A$ the universal cover of A of course $\tilde{A} = [0, 1] \times \mathbf{R}$ and if we denote by T a generator of the group of deck transformation of $\tilde{A} \rightarrow A$, we can write under this identification $T(x) = x + 1$ where addition is to be taken in the \mathbf{R} coordinate. The map \bar{h} lifts to a proper map \tilde{h} which verifies $\tilde{h}T = T^{-1}\tilde{h}$. It follows that \tilde{h} can be extended to the compactification of \tilde{A} by its two ends $\varepsilon_-, \varepsilon_+$ by a map which exchange these to ends. Since $\tilde{A} \cup \{\varepsilon_-, \varepsilon_+\}$ is homeomorphic to a disk \tilde{h} has a non empty compact set \tilde{F}_1 of fixed points which does not intersect the boundary because \tilde{h} exchange ε_- and ε_+ and \bar{h} has no fixed point on the boundary of A . Remark that the index of \tilde{F}_1 is 1. Moreover, the map $\tilde{A} \rightarrow A$ is injective on \tilde{F}_1 because if $\tilde{h}(x) = x$ we have $\tilde{h}(x+n) = x - n \neq x + n$ if $n \neq 0$. Since $\tilde{A} \rightarrow A$ is a covering it is clear that this map is also injective in a neighborhood of \tilde{F}_1 . It follows that the image F_1 of \tilde{F}_1 under $\tilde{A} \rightarrow A$ is a compact non empty set of fixed points of \bar{h} which has index 1. If $x \in \tilde{F}_1$, we have $T\tilde{h}(x+n) = T(x-n) = x + 1 - n \neq x + n$ for all n because $1/2 \notin \mathbf{Z}$. It follows that F_2 , the image under $\tilde{A} \rightarrow A$ of $\text{Fix}(T\tilde{h})$ —which is also a compact non empty set of fixed points of \bar{h} with index 1—is disjoint from F_1 . If $x \in A$ is a fixed point of \bar{h} , it lifts to a point $\tilde{x} \in \tilde{A}$ which verifies $\tilde{h}(\tilde{x}) = \tilde{x} + n$. If $n = 2k$ then $\tilde{h}(\tilde{x}+k) = \tilde{h}(\tilde{x}) - k = \tilde{x} + 2k - k = \tilde{x} + k$. If $n = 2k - 1$ then $T\tilde{h}(\tilde{x}+k) = T(\tilde{x}+2k-1-k) = \tilde{x} + k$. This shows clearly that $\text{Fix}(\bar{h}) = F_1 \cup F_2$. Since \bar{h} is equal to h in a neighborhood of $\text{Fix}(\bar{h}) = \text{Fix}(h)$, this ends the proof. \square

If we combine the Main Lemma 1.1 and lemma 2.1, we obtain:

LEMMA 2.2. *Let $h: \mathbf{S}^2 \rightarrow \mathbf{S}^2$ be an orientation preserving homeomorphism. If h has a non wandering point which is not a fixed point, then the set $\text{Fix}(h)$ can be written as a disjoint union $\text{Fix}(h) = F_1 \cup F_2$ with F_1 and F_2 closed non empty and having fixed point index equal to 1.*

Since we can compactify an orientation preserving homeomorphism of \mathbf{R}^2 by an orientation preserving homeomorphism of \mathbf{S}^2 with one more fixed point at infinity, we obtain the next two corollaries.

COROLLARY 2.3. (Brouwer's Lemma on translation arcs). *Let $h: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a fixed point free orientation preserving homeomorphism. Then h has no periodic point, each point wanders under h . Moreover, if α is a translation arc, the union $\bigcup_{n \in \mathbf{Z}} h^n(\alpha)$ is homeomorphic to a line and it does not accumulate on itself.*

COROLLARY 2.4. *Let $h: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be an orientation preserving homeomorphism. If the non wandering set of h is not reduced to the set of fixed points then there is a compact non empty subset $F \subset \text{Fix}(h)$ which has fixed point index equal to 1.*