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A. FATHI

1. An isotopy closing lemma

We will prove the following closing lemma:

MAIN LEMMA 1.1. Let $h: M \to M$ be a homeomorphism of the connected manifold M. If h has a nonwandering point which is not a fixed point, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:

- (i) $h_0 = h;$
- (ii) $h_t = h$ outside a compact subset of $M \setminus Fix(h)$ which does not depend on t;
- (iii) Fix $(h_t) = Fix (h)$;
- (iv) h_1 has a periodic point of period 2 in $M \setminus Fix(h_1)$.

We will need several elementary lemmas. The first lemma is easy.

LEMMA 1.2. Let $\varphi_1, ..., \varphi_k$ be homeomorphisms of a space Z. If X is a subset of Z, we have

$$\varphi_k \dots \varphi_1(X) \subset X \cup \left(\bigcup_{i=1}^k \operatorname{supp}(\varphi_i)\right).$$

LEMMA 1.3. Suppose that h and $\phi_1, ..., \phi_k$ are homeomorphisms of the space Y. If we have

$$\forall i = 1, ..., k, h(\operatorname{supp} \varphi_i) \cap (\bigcup_{j \leq i} \operatorname{supp} \varphi_j) = \emptyset$$

then Fix $(\varphi_k \dots \varphi_1 h) = Fix(h)$.

Proof. Since $h(\operatorname{supp} \varphi_i) \cap \operatorname{supp} \varphi_i = \emptyset$, we have

Fix
$$(h) \cap (\bigcup_{i=1}^{k} \operatorname{supp} \varphi_i) = \emptyset$$
.

This implies the inclusion Fix $(h) \subset$ Fix $(\varphi_k \dots \varphi_1 h)$.

We prove the other inclusion by induction on k.

Suppose k = 1. If $\varphi_1 h(x) = x$ and $h(x) \neq x$ then certainly $h(x) \in \text{supp } \varphi_1$ and hence also $x = \varphi_1 h(x) \in \text{supp } \varphi_1$. But this is impossible, since $h(\text{supp } \varphi_1)$ $\cap \text{supp } \varphi_1 = \emptyset$.

Suppose the lemma true for k - 1. Let x be such that $\varphi_k \varphi_{k-1} \dots \varphi_1 h(x) = x$. This is equivalent to $\varphi_{k-1} \dots \varphi_1 h(x) = \varphi_k^{-1}(x)$. If $x \notin \text{supp } \varphi_k$, we obtain $\varphi_{k-1} \dots \varphi_1 h(x) = x$. By the induction hypothesis, this gives $x \in \text{Fix}(h)$. If $x \in \text{supp } \varphi_k$, then $h(x) \in h(\text{supp } \varphi_k)$, and since $h(x) = \varphi_1^{-1} \dots \varphi_{k-1}^{-1} \varphi_k^{-1}(x)$, we obtain, by 1.2, that $h(x) \in \bigcup_{j \leq k} \text{supp } \varphi_j$ which is disjoint from $h(\text{supp } \varphi_k)$!

The next definition is due to Brouwer.

Definition 1.4 (Translation arc). Let $h: Z \to Z$ be a homeomorphism of the space Z. An injective arc $\alpha \subset Z$ is called a translation arc (for h) if α joins some point x to its image h(x) and $h(\alpha) \cap \dot{\alpha} = \emptyset$, where $\dot{\alpha}$ is α minus its extremities. Remark that α does not contain any of the fixed points of h. Moreover, we have $h(x) \in \alpha \cap h(\alpha)$ and if $\alpha \cap h(\alpha) \neq \{h(x)\}$ then $x = h^2(x)$.

LEMMA 1.5 (Brouwer). Let $h: M \to M$ be a homeomorphism of the manifold M. If y and h(y) are contained in the same component of $M \setminus Fix(h)$, then there exists a translation arc α with $y \in \dot{\alpha}$.

Proof (Well known). We can assume M connected and Fix $(h) = \emptyset$. Let B be a subset of M homeomorphic to the euclidean closed ball of the same dimension as M, containing y in its interior and with h(B) $\cap B = \emptyset$. Since M is connected, there exists an isotopy $\{\theta_t \mid t \in [0, 1]\}$ such that $\theta_0 = \text{Id}, \theta_t(y) = y$ and $\theta_1(h(y)) \in B$. If we put $B_t = \theta_t^{-1}(B)$, there is a first t such that $B_t \cap h(B_t) \neq \emptyset$, we call s this first t. We have:

- (i) y is in the interior of B_s ;
- (ii) the interiors of B_s and $h(B_s)$ are disjoint;
- (iii) B_s intersects $h(B_s)$ in a point which on the boundary of each one of them. If we call h(x) this point, then x is also in the boundary of B_s .

It follows that we can find an arc $\alpha \subset B_s$ between x and h(x), with $\dot{\alpha}$ contained in the interior of B_s . By (ii) above, $h(\alpha) \cap \dot{\alpha} = \emptyset$.

PROPOSITION 1.6. Let α be a translation arc for the homeomorphism h of the connected manifold M. If for some $n \ge 2$ we have $h^n(\alpha) \cap \alpha \neq \emptyset$, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:

- (i) $h_0 = h;$
- (ii) $h_t = h$ outside a compact subset of $M \setminus Fix(h)$ which does not depend on t;
- (iii) Fix $(h_t) = Fix (h)$;
- (iv) h_1 has a periodic point of period 2 in $M \setminus Fix(h_1)$.

Proof. We call x, h(x) the extremities of α . By 1.4, we are reduced to the case $\alpha \cap h(\alpha) = \{h(x)\}$. Call n + 1 the first integer ≥ 2 such that $h^{n+1}(\alpha) \cap \alpha \ne \emptyset$. Let $z \in h^{n+1}(\alpha) \cap \alpha$. By our choice of n + 1, if $n + 1 \ge 3$ and the fact that $\alpha \cap h(\alpha) = \{h(x)\}$, if n + 1 = 2, we have $z \ne h(x)$. We orient the injective segment $\bigcup_{i=0}^{n} h^{i}(\alpha)$ from x to $h^{n+1}(x)$. We denote by \le the natural order induced by this orientation with x < h(x). We first consider the case where $h^{-2}(z) \le z$. Let $\beta \subset \alpha \setminus \{h(x)\}$ be the compact sub arc joining $h^{-2}(z)$ to z. We have $h(\beta) \cap \beta = \emptyset$. Let V be a small connected neighborhood of β such that $h(V) \cap V = \emptyset$. Call φ_t an isotopy of M with compact support contained in V and such that $\varphi_0 = \text{Id}$ and $\varphi_1(z) = h^{-2}(z)$. We can define h_t as $\varphi_t h$. By 1.3, the conditions $h(V) \cap V = \emptyset$ and $\supp(\varphi_t) \subset V \text{ imply that Fix } (\varphi_t h) = \text{Fix } (h)$. Furthermore, since $h^{-2}(z) \in V$, we have $h^{-1}(z) \in h(V)$ which does not intersect $\supp(\varphi_1)$. It follows that $(\varphi_1 h) (h^{-2}(z)) = h^{-1}(z)$, and hence we obtain $(\varphi_1 h)^2 (h^{-2}(z)) = \varphi_1(z) = h^{-2}(z)$.

We now consider the case $z \leq h^{-2}(z)$. We choose $z_0 = z \leq z_1 \leq ... \leq z_k$ $= h^{-2}(z)$ in the segment $\bigcup_{i=0}^{n-1} h^i(\alpha)$ such that the subsegment $[z_0, z_i]$ is disjoint from the image $h([z_{i-1}, z_i])$, for i = 1, ..., k. We can find neighborhoods $V_1, ..., V_i, ..., V_k$ of $[z_0, z_1], ..., [z_{i-1}, z_i], ..., [z_{k-1}, z_k]$ such that $h(V_i) \cap (\bigcup_{j \leq i} V_j) = \emptyset$. It is easy to construct a sequence of isotopies with compact support $\varphi_i^1, ..., \varphi_i^k$ such that $\varphi_1^i(z_{i-1}) = z_i$ and supp $\varphi_i^i \subset V_i$. By 1.3, this last condition and the fact that $h(V_i) \cap (\bigcup_{j \leq i} V_j) = \emptyset$, for i = 1, ..., k, imply the equality Fix $(\varphi_i^k ... \varphi_i^1 h) =$ Fix (h). Moreover, since $h^{-1}(z) \in h(V_k)$ which is disjoint from $\bigcup_{i=1}^k$ supp φ_i , we have $(\varphi_1^k ... \varphi_1^1 h)^2 (h^{-2}(z)) = h^{-2}(z)$.

COROLLARY 1.7. Let α be a translation arc for the homeomorphism h of the connected manifold M. Suppose that some point of α is in the closure of $\bigcup_{n \ge 2} h^n(\alpha)$, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:

- (i) $h_0 = h;$
- (ii) $h_t = h$ outside a compact subset of $M \setminus Fix(h)$;
- (iii) Fix $(h_t) = Fix (h)$;
- (iv) h_1 has a periodic point of period 2 in $M \setminus Fix(h_1)$.

Proof. We can suppose that $\alpha \cap (\bigcup_{n \ge 2} h^n(\alpha)) = \emptyset$. Then we will find an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:

- (i) $h_0 = h;$
- (ii) α is a translation arc for each h_t ;
- (iii) Fix $(h_t) = Fix(h);$

(iv) $h_1^n(\alpha) \cap \alpha \neq \emptyset$, for some $n \ge 2$;

(v) $h_t = h$ outside a compact subset of M which does not depend on t. It will then suffice to apply proposition 1.6 to h_1 .

We denote by x and h(x) the extremities of α . Let us call $z \in \alpha \setminus \{h(x)\}$ a point of accumulation of $\bigcup_{n \ge 2} h^n(\alpha)$. Let V be a small connected neighborhood of z which does not intersect $h(\alpha)$. Let $n \ge 2$ be the first integer such that $h^n(\alpha)$ intersects V. We can find an isotopy $\{\varphi_t \mid t \in [0, 1]\}$, with compact support contained in V, such that $\varphi_0 = \text{Id}$ and $\varphi_1 h^n(\alpha) \ge z$. It suffices to define h_t as $\varphi_t h$.

LEMMA 1.8. Let h be a homeomorphism of the manifold M. Suppose that h has a non-wandering point for h which is not a fixed point, then there exists an isotopy $\{h_t \mid t \in [0, 1]\}$ such that:

- (i) $h_0 = h;$
- (ii) $h_t = h$ outside a compact subset of $M \setminus Fix(h)$;
- (iii) Fix $(h_t) = Fix (h)$;
- (iv) there is a periodic point of h_1 which is not a fixed point.

Proof. Call z a non-wandering point which is not a fixed point. Let V be a small open connected neighborhood of z such that $h(V) \cap V = \emptyset$. Call $n \ge 2$ the first integer such that $h^n(V) \cap V \ne \emptyset$. Choose $y \in h^{-n}(V) \cap V \ne \emptyset$. Call $\{\varphi_t \mid t \in [0, 1]\}$ an isotopy with compact support in V and such that $\varphi_1(h^n(y)) = y$. It suffices to put $h_t = \varphi_t h$.

Proof of the Main Lemma. If h leaves invariant each component of $M \setminus Fix(h)$, the Main Lemma follows from what we have done. If this is not the case then by a result of Brown and Kister [BK] $M \setminus Fix(h)$ has exactly two connected components which are exchanged by h. It is easy to construct the required isotopy in this case.

Remarks 1.9. (i) In the proof of the Main Lemma, we use the Brown-Kister result only in the case where Fix (h) disconnects x from h(x). In particular, if M is connected, of dimension ≥ 2 , and if Fix (h) is finite we do not have to use it.

(ii) It follows from [Bw2, Lemma 6.3] that a homeomorphism of a connected manifold of dimension ≥ 3 which is not the identity can be isotoped without changing the set of fixed point to a homeomorphism with a periodic point of period 2. Hence, the main lemma 1.1 is useful only for dimension 2.