

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 33 (1987)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** AN ORBIT CLOSING PROOF OF BROUWER'S LEMMA ON TRANSLATION ARCS  
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**Kapitel:** 1. An isotopy closing lemma  
**DOI:** <https://doi.org/10.5169/seals-87901>

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## 1. AN ISOTOPY CLOSING LEMMA

We will prove the following closing lemma:

**MAIN LEMMA 1.1.** *Let  $h: M \rightarrow M$  be a homeomorphism of the connected manifold  $M$ . If  $h$  has a nonwandering point which is not a fixed point, then there exists an isotopy  $\{h_t \mid t \in [0, 1]\}$  such that:*

- (i)  $h_0 = h$ ;
- (ii)  $h_t = h$  outside a compact subset of  $M \setminus \text{Fix}(h)$  which does not depend on  $t$ ;
- (iii)  $\text{Fix}(h_t) = \text{Fix}(h)$ ;
- (iv)  $h_1$  has a periodic point of period 2 in  $M \setminus \text{Fix}(h_1)$ .

We will need several elementary lemmas. The first lemma is easy.

**LEMMA 1.2.** *Let  $\varphi_1, \dots, \varphi_k$  be homeomorphisms of a space  $Z$ . If  $X$  is a subset of  $Z$ , we have*

$$\varphi_k \dots \varphi_1(X) \subset X \cup \left( \bigcup_{i=1}^k \text{supp}(\varphi_i) \right).$$

**LEMMA 1.3.** *Suppose that  $h$  and  $\varphi_1, \dots, \varphi_k$  are homeomorphisms of the space  $Y$ . If we have*

$$\forall i = 1, \dots, k, h(\text{supp} \varphi_i) \cap \left( \bigcup_{j \leq i} \text{supp} \varphi_j \right) = \emptyset$$

*then  $\text{Fix}(\varphi_k \dots \varphi_1 h) = \text{Fix}(h)$ .*

*Proof.* Since  $h(\text{supp} \varphi_i) \cap \text{supp} \varphi_i = \emptyset$ , we have

$$\text{Fix}(h) \cap \left( \bigcup_{i=1}^k \text{supp} \varphi_i \right) = \emptyset.$$

This implies the inclusion  $\text{Fix}(h) \subset \text{Fix}(\varphi_k \dots \varphi_1 h)$ .

We prove the other inclusion by induction on  $k$ .

Suppose  $k = 1$ . If  $\varphi_1 h(x) = x$  and  $h(x) \neq x$  then certainly  $h(x) \in \text{supp} \varphi_1$  and hence also  $x = \varphi_1 h(x) \in \text{supp} \varphi_1$ . But this is impossible, since  $h(\text{supp} \varphi_1) \cap \text{supp} \varphi_1 = \emptyset$ .

Suppose the lemma true for  $k - 1$ . Let  $x$  be such that  $\varphi_k \varphi_{k-1} \dots \varphi_1 h(x) = x$ . This is equivalent to  $\varphi_{k-1} \dots \varphi_1 h(x) = \varphi_k^{-1}(x)$ . If  $x \notin \text{supp} \varphi_k$ , we obtain  $\varphi_{k-1} \dots \varphi_1 h(x) = x$ . By the induction hypothesis, this gives  $x \in \text{Fix}(h)$ . If

$x \in \text{supp } \varphi_k$ , then  $h(x) \in h(\text{supp } \varphi_k)$ , and since  $h(x) = \varphi_1^{-1} \dots \varphi_{k-1}^{-1} \varphi_k^{-1}(x)$ , we obtain, by 1.2, that  $h(x) \in \bigcup_{j \leq k} \text{supp } \varphi_j$  which is disjoint from  $h(\text{supp } \varphi_k)$ !  $\square$

The next definition is due to Brouwer.

*Definition 1.4 (Translation arc).* Let  $h: Z \rightarrow Z$  be a homeomorphism of the space  $Z$ . An injective arc  $\alpha \subset Z$  is called a translation arc (for  $h$ ) if  $\alpha$  joins some point  $x$  to its image  $h(x)$  and  $h(\alpha) \cap \dot{\alpha} = \emptyset$ , where  $\dot{\alpha}$  is  $\alpha$  minus its extremities. Remark that  $\alpha$  does not contain any of the fixed points of  $h$ . Moreover, we have  $h(x) \in \alpha \cap h(\alpha)$  and if  $\alpha \cap h(\alpha) \neq \{h(x)\}$  then  $x = h^2(x)$ .

**LEMMA 1.5 (Brouwer).** Let  $h: M \rightarrow M$  be a homeomorphism of the manifold  $M$ . If  $y$  and  $h(y)$  are contained in the same component of  $M \setminus \text{Fix}(h)$ , then there exists a translation arc  $\alpha$  with  $y \in \dot{\alpha}$ .

*Proof (Well known).* We can assume  $M$  connected and  $\text{Fix}(h) = \emptyset$ . Let  $B$  be a subset of  $M$  homeomorphic to the euclidean closed ball of the same dimension as  $M$ , containing  $y$  in its interior and with  $h(B) \cap B = \emptyset$ . Since  $M$  is connected, there exists an isotopy  $\{\theta_t \mid t \in [0, 1]\}$  such that  $\theta_0 = \text{Id}$ ,  $\theta_t(y) = y$  and  $\theta_1(h(y)) \in B$ . If we put  $B_t = \theta_t^{-1}(B)$ , there is a first  $t$  such that  $B_t \cap h(B_t) \neq \emptyset$ , we call  $s$  this first  $t$ . We have:

- (i)  $y$  is in the interior of  $B_s$ ;
- (ii) the interiors of  $B_s$  and  $h(B_s)$  are disjoint;
- (iii)  $B_s$  intersects  $h(B_s)$  in a point which on the boundary of each one of them. If we call  $h(x)$  this point, then  $x$  is also in the boundary of  $B_s$ .

It follows that we can find an arc  $\alpha \subset B_s$  between  $x$  and  $h(x)$ , with  $\dot{\alpha}$  contained in the interior of  $B_s$ . By (ii) above,  $h(\alpha) \cap \dot{\alpha} = \emptyset$ .  $\square$

**PROPOSITION 1.6.** Let  $\alpha$  be a translation arc for the homeomorphism  $h$  of the connected manifold  $M$ . If for some  $n \geq 2$  we have  $h^n(\alpha) \cap \alpha \neq \emptyset$ , then there exists an isotopy  $\{h_t \mid t \in [0, 1]\}$  such that:

- (i)  $h_0 = h$ ;
- (ii)  $h_t = h$  outside a compact subset of  $M \setminus \text{Fix}(h)$  which does not depend on  $t$ ;
- (iii)  $\text{Fix}(h_t) = \text{Fix}(h)$ ;
- (iv)  $h_1$  has a periodic point of period 2 in  $M \setminus \text{Fix}(h_1)$ .

*Proof.* We call  $x, h(x)$  the extremities of  $\alpha$ . By 1.4, we are reduced to the case  $\alpha \cap h(\alpha) = \{h(x)\}$ . Call  $n + 1$  the first integer  $\geq 2$  such that  $h^{n+1}(\alpha) \cap \alpha \neq \emptyset$ . Let  $z \in h^{n+1}(\alpha) \cap \alpha$ . By our choice of  $n + 1$ , if  $n + 1 \geq 3$  and the fact that  $\alpha \cap h(\alpha) = \{h(x)\}$ , if  $n + 1 = 2$ , we have  $z \neq h(x)$ . We orient the injective segment  $\bigcup_{i=0}^n h^i(\alpha)$  from  $x$  to  $h^{n+1}(x)$ . We denote by  $\leq$  the natural order induced by this orientation with  $x < h(x)$ . We first consider the case where  $h^{-2}(z) \leq z$ . Let  $\beta \subset \alpha \setminus \{h(x)\}$  be the compact sub arc joining  $h^{-2}(z)$  to  $z$ . We have  $h(\beta) \cap \beta = \emptyset$ . Let  $V$  be a small connected neighborhood of  $\beta$  such that  $h(V) \cap V = \emptyset$ . Call  $\varphi_t$  an isotopy of  $M$  with compact support contained in  $V$  and such that  $\varphi_0 = \text{Id}$  and  $\varphi_1(z) = h^{-2}(z)$ . We can define  $h_t$  as  $\varphi_t h$ . By 1.3, the conditions  $h(V) \cap V = \emptyset$  and  $\text{supp}(\varphi_t) \subset V$  imply that  $\text{Fix}(\varphi_t h) = \text{Fix}(h)$ . Furthermore, since  $h^{-2}(z) \in V$ , we have  $h^{-1}(z) \in h(V)$  which does not intersect  $\text{supp}(\varphi_1)$ . It follows that  $(\varphi_1 h)(h^{-2}(z)) = h^{-1}(z)$ , and hence we obtain  $(\varphi_1 h)^2(h^{-2}(z)) = \varphi_1(z) = h^{-2}(z)$ .

We now consider the case  $z \leq h^{-2}(z)$ . We choose  $z_0 = z \leq z_1 \leq \dots \leq z_k = h^{-2}(z)$  in the segment  $\bigcup_{i=0}^{n-1} h^i(\alpha)$  such that the subsegment  $[z_0, z_i]$  is disjoint from the image  $h([z_{i-1}, z_i])$ , for  $i = 1, \dots, k$ . We can find neighborhoods  $V_1, \dots, V_i, \dots, V_k$  of  $[z_0, z_1], \dots, [z_{i-1}, z_i], \dots, [z_{k-1}, z_k]$  such that  $h(V_i) \cap (\bigcup_{j \leq i} V_j) = \emptyset$ . It is easy to construct a sequence of isotopies with compact support  $\varphi_1^1, \dots, \varphi_1^k$  such that  $\varphi_1^i(z_{i-1}) = z_i$  and  $\text{supp} \varphi_1^i \subset V_i$ . By 1.3, this last condition and the fact that  $h(V_i) \cap (\bigcup_{j \leq i} V_j) = \emptyset$ , for  $i = 1, \dots, k$ , imply the equality  $\text{Fix}(\varphi_1^k \dots \varphi_1^1 h) = \text{Fix}(h)$ . Moreover, since  $h^{-1}(z) \in h(V_k)$  which is disjoint from  $\bigcup_{i=1}^k \text{supp} \varphi_i$ , we have  $(\varphi_1^k \dots \varphi_1^1 h)^2(h^{-2}(z)) = h^{-2}(z)$ .  $\square$

**COROLLARY 1.7.** *Let  $\alpha$  be a translation arc for the homeomorphism  $h$  of the connected manifold  $M$ . Suppose that some point of  $\alpha$  is in the closure of  $\bigcup_{n \geq 2} h^n(\alpha)$ , then there exists an isotopy  $\{h_t \mid t \in [0, 1]\}$  such that:*

- (i)  $h_0 = h$ ;
- (ii)  $h_t = h$  outside a compact subset of  $M \setminus \text{Fix}(h)$ ;
- (iii)  $\text{Fix}(h_t) = \text{Fix}(h)$ ;
- (iv)  $h_1$  has a periodic point of period 2 in  $M \setminus \text{Fix}(h_1)$ .

*Proof.* We can suppose that  $\alpha \cap (\bigcup_{n \geq 2} h^n(\alpha)) = \emptyset$ . Then we will find an isotopy  $\{h_t \mid t \in [0, 1]\}$  such that:

- (i)  $h_0 = h$ ;
- (ii)  $\alpha$  is a translation arc for each  $h_t$ ;
- (iii)  $\text{Fix}(h_t) = \text{Fix}(h)$ ;

- (iv)  $h_1^n(\alpha) \cap \alpha \neq \emptyset$ , for some  $n \geq 2$ ;
  - (v)  $h_t = h$  outside a compact subset of  $M$  which does not depend on  $t$ .
- It will then suffice to apply proposition 1.6 to  $h_1$ .

We denote by  $x$  and  $h(x)$  the extremities of  $\alpha$ . Let us call  $z \in \alpha \setminus \{h(x)\}$  a point of accumulation of  $\bigcup_{n \geq 2} h^n(\alpha)$ . Let  $V$  be a small connected neighborhood of  $z$  which does not intersect  $h(\alpha)$ . Let  $n \geq 2$  be the first integer such that  $h^n(\alpha)$  intersects  $V$ . We can find an isotopy  $\{\varphi_t \mid t \in [0, 1]\}$ , with compact support contained in  $V$ , such that  $\varphi_0 = \text{Id}$  and  $\varphi_1 h^n(\alpha) \ni z$ . It suffices to define  $h_t$  as  $\varphi_t h$ .  $\square$

**LEMMA 1.8.** *Let  $h$  be a homeomorphism of the manifold  $M$ . Suppose that  $h$  has a non-wandering point for  $h$  which is not a fixed point, then there exists an isotopy  $\{h_t \mid t \in [0, 1]\}$  such that:*

- (i)  $h_0 = h$ ;
- (ii)  $h_t = h$  outside a compact subset of  $M \setminus \text{Fix}(h)$ ;
- (iii)  $\text{Fix}(h_t) = \text{Fix}(h)$ ;
- (iv) *there is a periodic point of  $h_1$  which is not a fixed point.*

*Proof.* Call  $z$  a non-wandering point which is not a fixed point. Let  $V$  be a small open connected neighborhood of  $z$  such that  $h(V) \cap V = \emptyset$ . Call  $n \geq 2$  the first integer such that  $h^n(V) \cap V \neq \emptyset$ . Choose  $y \in h^{-n}(V) \cap V \neq \emptyset$ . Call  $\{\varphi_t \mid t \in [0, 1]\}$  an isotopy with compact support in  $V$  and such that  $\varphi_1(h^n(y)) = y$ . It suffices to put  $h_t = \varphi_t h$ .  $\square$

*Proof of the Main Lemma.* If  $h$  leaves invariant each component of  $M \setminus \text{Fix}(h)$ , the Main Lemma follows from what we have done. If this is not the case then by a result of Brown and Kister [BK]  $M \setminus \text{Fix}(h)$  has exactly two connected components which are exchanged by  $h$ . It is easy to construct the required isotopy in this case.  $\square$

**Remarks 1.9.** (i) In the proof of the Main Lemma, we use the Brown-Kister result only in the case where  $\text{Fix}(h)$  disconnects  $x$  from  $h(x)$ . In particular, if  $M$  is connected, of dimension  $\geq 2$ , and if  $\text{Fix}(h)$  is finite we do not have to use it.

(ii) It follows from [Bw2, Lemma 6.3] that a homeomorphism of a connected manifold of dimension  $\geq 3$  which is not the identity can be isotoped without changing the set of fixed point to a homeomorphism with a periodic point of period 2. Hence, the main lemma 1.1 is useful only for dimension 2.